Physics 610 Homework 7

Due Monday 12 November

1 Spinors

In class we found explicit expressions for the spinors $u_{p,s}$ and $v_{p,s}$ for the special case that $p^\mu = (E, 0, 0, p_z)$ ($E^2 = p_z^2 + m^2$ as usual) and $s = \pm\frac{1}{2}$ along the z axis.

Verify that these explicit expressions obey the relations

$$\begin{aligned}
\bar{u}_{p,s} u_{p,s'} &= 2m\delta_{ss'}\,, && (1)\\
\bar{v}_{p,s} v_{p,s'} &= -2m\delta_{ss'}\,, && (2)\\
\bar{u}_{p,s} v_{p,s'} &= \bar{v}_{p,s} u_{p,s'} = 0\,, && (3)\\
u^\dagger_{p,s} u_{p,s'} &= 2E\delta_{ss'}\,, && (4)\\
\bar{u}_{p,s}\gamma^\mu u_{p,s'} &= 2p^\mu\delta_{ss'}\,, && (5)\\
\sum_s u_{p,s}\bar{u}_{p,s} &= -\not{p} + m\,, && (6)\\
\sum_s v_{p,s}\bar{v}_{p,s} &= -\not{p} - m\,. && (7)
\end{aligned}$$

2 Time ordered correlation function

In class we found that, for the free theory of Dirac spinors,

$$S^{>}_{ab}(x-y) \equiv +i\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\Big(-\not{p}_{ab} + m\delta_{ab}\Big) \times i(2\pi)\delta(p^2+m^2)\Theta(p^0)\,. \tag{8}$$

Find the similar expression for (note signs!)

$$S^{<}_{ab}(x-y) \equiv -i\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle\,.$$

Then use the definition of the time-ordered correlation function,

$$S(x) \equiv S^{>}(x)\Theta(x^0) + S^{<}(x)\Theta(-x^0) \tag{9}$$

and the integral representation of $\Theta(x^0)$, to derive the expression

$$S(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\,\frac{-\not{p} + m}{p^2 + m^2 - i\epsilon}\,. \tag{10}$$

Hint: the derivation is similar to the one we encountered in studying the scalar propagator $\Delta_0(x)$. At some point you may want to change variables in one term from p to $-p$.

3 Conserved charge

Verify that, if a Dirac field has charge +1 under a U(1) symmetry as in the last homework, so the Nöther current is $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, that the charge operator is given by

$$Q = \sum_{p,s} \frac{1}{2E_p} \left(b^\dagger_{p,s} b_{p,s} - d^\dagger_{p,s} d_{p,s} \right) \tag{11}$$

up to an irrelevant constant.

4 Practice with Grassmann numbers

Let Θ_i ($i = 1, 2, 3, \ldots$) be Grassmann numbers; Roman letters a, b, c refer to ordinary (complex) numbers. Simplify each of the following expressions. Your final expression should be a polynomial, where each term has the Θ_i appear in order, that is, $\Theta_1\Theta_2\Theta_3$ not $\Theta_3\Theta_1\Theta_2$.

$$(a + b\Theta_1 + c\Theta_1\Theta_2)(r\Theta_3 + s\Theta_2\Theta_3 + t\Theta_2\Theta_4) = \tag{12}$$

$$\exp\Big(1 + \Theta_1 + \Theta_2 - c\Theta_1\Theta_3\Big) = \tag{13}$$

$$\frac{\partial}{\partial\Theta_1}\Big(c_1 + c_2\Theta_1 + c_2\Theta_2 + c_3\Theta_1\Theta_2\Theta_3\Big) + \frac{\partial}{\partial\Theta_3}\Big(d_1\Theta_2 + d_2\Theta_1\Theta_2\Theta_3 + d_3\Theta_2\Theta_3\Theta_4\Big) = \tag{14}$$

$$\int d\Theta_2\Big(a + b\Theta_1 + c\Theta_2 + d\Theta_1\Theta_2 + e\Theta_2\Theta_3\Theta_4\Big) = \tag{15}$$

$$\int d\Theta_1 d\Theta_2\ \exp\Big(r + c_1\Theta_1 + c_2\Theta_2 + c_3\Theta_1\Theta_2\Big) = \tag{16}$$