Physics 610 Homework 8

Due Monday 19 November 2012

1 Complete Set of Grassmann States

For Θ_i, $\bar{\Theta}_i$, Θ'_i, $\bar{\Theta}'_i$ each independent n-member sets of Grassmann variables, and using the summation convention $\bar{\Theta}\Theta \equiv \bar{\Theta}_i\Theta_i \equiv \sum_i \bar{\Theta}_i\Theta_i$, prove the identity

$$e^{\bar{\Theta}\Theta} = \int \prod_i d\Theta'_i d\bar{\Theta}'_i \, e^{\bar{\Theta}\Theta'} e^{\bar{\Theta}'\Theta'} e^{-\bar{\Theta}'\Theta} \,. \tag{1}$$

Hint: First show that it is true when $n = 1$ (brute force?). Then show that, for larger n, it factorizes into n copies of the $n = 1$ case.

2 Moving past exponents

We use the same notation as in the previous problem. Suppose that A is a polynomial in Grassmann variables which is independent of $\bar{\Theta}$, that is, $\frac{\partial}{\partial\bar{\Theta}_i}A = 0$ for all i. Suppose also that B is a polynomial in Grassmann variables which does not depend on Θ, so $\frac{\partial}{\partial\Theta_i}B = 0$ for all i.

First, show that

$$Be^{\bar{\Theta}\Theta}\Theta_i A = B\frac{\partial}{\partial\bar{\Theta}_i}e^{\bar{\Theta}\Theta}A \,. \tag{2}$$

Which conditions on A, B did you need? [The exponent is a sum on all the Θ, but next to A stands just one specific Θ.]

Next, show that

$$\int \prod_i d\Theta_i \; Be^{\bar{\Theta}\Theta}\frac{\partial}{\partial\Theta_j}A = \int \prod_i d\Theta_i \; B\bar{\Theta}_j e^{\bar{\Theta}\Theta}A \,. \tag{3}$$

[Again all Θ are integrated over, but the derivative acting on A is just one of the Θ's.] Which conditions on A, B did you need in this case?

3 Change of variables

For ordinary (commuting) numbers, if we make a change of variables

$$x'_i = R_{ij}x_j \qquad \text{so} \qquad x_i = R^{-1}_{ij}x'_j \tag{4}$$

with R_{ij} a nonsingular matrix, we usually define partial derivatives in terms of the new variables as

$$\frac{\partial}{\partial x'_i} x'_j = \delta_{ij}\,, \tag{5}$$

so that

$$\frac{\partial}{\partial x_i} x'_j = R_{ji}\,, \qquad \frac{\partial}{\partial x'_i} x_j = R^{-1}_{ji}\,. \tag{6}$$

Define changes of variables and differentiation with respect to the new variables in the same way for Grassmann numbers:

$$\Theta'_i = R_{ij}\Theta_j \qquad \text{so} \qquad \Theta_i = R^{-1}_{ij}\Theta'_j\,. \tag{7}$$

Use the fact that

$$\int d\Theta_n \ldots d\Theta_1 \; \Theta_1 \ldots \Theta_n = 1 \tag{8}$$

and the fact that $\int d\Theta = \frac{\partial}{\partial\Theta}$ to prove that

$$\int d\Theta'_n \ldots d\Theta'_1 \; \Theta_1 \ldots \Theta_n = \mathrm{Det}\, R^{-1}\,. \tag{9}$$

Argue that, as a consequence,

$$\int d\Theta_n \ldots d\Theta_1 \; A = \mathrm{Det}\, R \int d\Theta'_n \ldots d\Theta'_1 \; A \tag{10}$$

where A is any polynomial in Grassmann variables. Therefore changes of variables in Grassmann variables give rise to Jacobians which are the inverses of those which arise for bosonic variables.

Hint: you may find it easiest to prove this by doing the case of one Grassmann variable and the case of two Grassmann variables explicitly, to see the pattern. Partial credit will be given if you can show these cases but you cannot generalize to larger numbers of variables.

4 Shifting integration variables

Show explicitly that, for the Grassmann variables $\bar{\Theta}$, Θ, η, and $\bar{\eta}$ (each a single variable, in this problem we will not deal with n-vectors of Grassmann numbers!)

$$\int d\Theta d\bar{\Theta} \; \exp\left(\bar{\Theta}\Theta + \bar{\eta}\Theta - \bar{\Theta}\eta\right) = \exp\left(\bar{\eta}\eta\right). \tag{11}$$

Hint: expand everything and use brute force.

5 Combining your results

Consider the integral

$$\int \prod_i d\Theta_i d\bar{\Theta}_i \exp\left(\bar{\Theta}_i M_{ij} \Theta_j + \bar{\eta}_i \Theta_i - \bar{\Theta}_i \eta_i\right). \tag{12}$$

Use the previous two results to prove that this integral equals

$$\text{Det}\, M \;\; \exp\left(\bar{\eta}_i M^{-1}_{ij} \eta_j\right). \tag{13}$$

Hint: consider defining $\Theta = M^{-1}\Theta'$. Then $\bar{\eta}\Theta = \bar{\eta}M^{-1}\Theta'$. Then define $\bar{\eta}' = \bar{\eta}M^{-1}$.