Physics 673 Homework 1

Due 16 January 2007

1 Free field theory

(a) It is well known that a free quantum field theory is related to an infinite number of quantum mechanical harmonic oscillators with different frequencies. You can view these harmonic oscillators as unit mass particles on springs with spring constants k, where k takes all values. Now imagine the following scenario: these particles also have non-zero charges, and I suspend the system in a constant electric field E_0. Ignoring the effects of any outgoing radiations due to the oscillations of charged particles, show that this system of harmonic oscillators is *equivalent* to a massive scalar field theory with a classical source term $\int d^4x \ J\varphi$ where J is the source and φ is the scalar field. Find the connection between E_0 and J. Also calculate the number of particles created in the field theory system if in the dual quantum mechanical system I switch on the electric field E_0 for a very short time Δt.

2 QED

An alternative way of regulating QED, rather than dimensional regularization, is called *Pauli-Villars* regularization. In this regularization, in addition to the ordinary electron of mass m, one introduces a "regulator" electron of mass Λ, with Λ much larger than any physical scale one needs to consider. The regulator field is given "wrong" statistics in the sense that there is not a -1 sign associated with a loop of the regulator field.

Calculate the 1-loop correction to the photon self-energy, and therefore to charge renormalization, in this regulator. Show that the result is finite when one adds the contributions of the normal and regulator electron, but there is a log dependence on the mass ratio Λ/m. What is the relation between e_0 and e_{meas}? Compare the result to dimensional regularization; in particular, find the relation between the regulator mass Λ and the dimensional regularization scale

$$\Lambda_{DR}\,, \qquad \ln\frac{\Lambda_{DR}^2}{\mu^2} \equiv \frac{1}{\epsilon} + \ln 4\pi - \gamma_E\,. \tag{1}$$

3 Scalar field

Consider the example of a scalar field $\varphi(x,t)$ in 1+1 dimensions with a Lagrangian that is slightly different from the standard one in the following way:

$$\mathcal{L} \;=\; \int dx \left[\frac{1}{2}(\nabla_0\varphi)^2 \;-\; \frac{1}{2}(\nabla_x\varphi)^2 - \frac{\lambda}{4}\left(\varphi^2 - \frac{m^2}{\lambda}\right)^2\right] \tag{2}$$

where λ and m are the parameters of the theory. Now using Eq. (2) answer the following questions:

1. Show that the system has two kinds of *classical* solutions: a trivial vacuum solution $\varphi(x) = \varphi_0 \equiv m/\sqrt{\lambda}$ and a non-trivial space dependent background solution $\varphi_{\rm cl}(x)$ with $\varphi_{\rm cl} \to -\varphi_0$ at $x \to -\infty$ and $\varphi_{\rm cl} \to +\varphi_0$ at $x \to +\infty$. Neither solution is time dependent. Quantize the small oscillations about these solutions and show that there is a tower of harmonic oscillator states exactly as one would have expected for a standard quantization of a field theory. To do this first define the small fluctuations in the following way:

   $$\varphi(x,t) \;=\; \varphi_{\rm cl}(x) \;+\; \eta(x,t) \tag{3}$$

   where $\eta(x,t)$ is the quantum field. Now split the (x,t) dependences of $\eta(x,t)$ in the following way:

   $$\eta(x,t) \;=\; \sum_i c_i(t)\;\eta_i(x) \tag{4}$$

   where – and this is crucial – $\eta_i(x)$ form the normal modes of oscillations in the system because we are allowing only small fluctuations about the minima of the potential. Show that these normal modes map to an equivalent *quantum-mechanical* system with a potential

   $$V(y) \;=\; 3\tanh^2 y \;-\; 1 \tag{5}$$

   y being proportional to x coordinate, and with only two discrete levels followed by a continuum of levels:

   $$\psi_0(y) \;=\; {\rm sech}^2 y,\;\; \psi_1(y) \;=\; \frac{\sinh\, y}{\cosh^2 y}, \qquad \psi_q(y) \;=\; e^{iqy}\left(3\tanh^2 y - 1 - q^2 - 3iq\,\tanh\, y\right) \tag{6}$$

   where $\psi_i(y)$ are the wavefunctions, $i = 0, 1, q$ with q forming the continuum levels. Compare this procedure with the standard quantization of a free scalar field theory.

2. Argue that one of the discrete levels you found above, for the nontrivial classical solution, corresponds to a harmonic oscillator with a "spring constant" of *zero*.

3. If we call the total zero-point energy from the tower of states of the first vacuum as E_1 and the total zero-point energy from the tower of states of the second non-trivial vacuum as E_2, then in general the difference

$$E_{\text{difference}} = E_2 - E_1 \tag{7}$$

is formally divergent. In standard QFT such divergent energies are usually ignored because we can always shift the potential energy by an arbitrary constant. However here we are comparing two different *sectors* of the same QFT, so we cannot ignore the divergent difference between the two zero-point energies. The analysis of this is particularly involved so we will go in steps. For this assignment show that $E_{\text{difference}}$ can at least be brought to the following form

$$E_{\text{difference}} \propto \int_{-\infty}^{+\infty} dk \frac{k\delta(k)}{\sqrt{k^2+2m}} + \mathcal{O}(\lambda) \tag{8}$$

where we have converted infinite sums to an integral using the standard procedure of $\sum\limits_{k_n} \to \frac{L}{2\pi}\int dk$ with a box size $L \to \infty$. The $\delta(k)$ factor appearing in Eq. (8) is a subtle phase factor. This may require some careful analysis to extract this. Show that in the limit $L \to \infty$, the non-zero phase factor is

$$\delta(k)\Big|_{L\to\infty} = -2\tan^{-1}\left(\frac{3\sqrt{2}km}{2m^2-2k^2}\right) \tag{9}$$

The next step is to regularize and renormalize the quantity by introducing counter-terms in the original Lagrangian. We will address these details later.

If you need help on this section, **BUG KESHAV**.

4 Yukawa theory

Let us consider Dirac fermion field ψ interacting with a real scalar field ϕ of mass M. The interaction term is

$$S_{\text{interaction}} = -i\int d^4x\; y\bar{\psi}\gamma^5\psi\phi \tag{10}$$

where y is a coupling constant.

Write the complete Lagrangian. Introduce all necessary counterterms. Compute all counterterms at the 1-loop level, keeping only the log divergent behavior (that is, do not attempt to find constants associated with 1-loop logarithmic divergences).