Physics 673 Homework 2

Due 6 February 2007

1 Playing with free scalar fields

Consider a free scalar field with propagator $D_F(x) \equiv \langle \mathrm{T}\{\phi(x)\phi(0)\}\rangle$.

1. Using results from last term's homeworks if convenient, show that the small x limit of $D_F(x)$ is divergent.

2. Define another field Φ as a linear combination of free fields $\phi(x)$ in the following way:

$$\Phi(y) = \int dx\; f(x)\phi(x+y)\,. \tag{1}$$

   Here $f(x)$ is a smooth, real function without singularities.

   Show that

$$\langle e^{\Phi}(0)\rangle = e^{\langle\Phi^2(0)\rangle/2} \tag{2}$$

   where powers of Φ should be viewed as field contractions. Can you also derive this result using path integrals?

3. Prove the following relation between time ordered contractions of the free scalar field $\phi(x)$ with $\phi(0)$:

$$\langle \mathrm{T} e^{i\phi(x)} e^{-i\phi(0)}\rangle = e^{D_F(x)-D_F(0)} \tag{3}$$

   Here you should consider the field ϕ to be smeared by some function f as above; the result should not depend on this smearing, only on the fact that the correlation functions remain Gaussian.

   (This result forms the underlying structure behind the spin-wave theory.)

2 Gross-Neveu model

In $1+1$ dimensions, Dirac fermions are 2-component and the (two) gamma matrices are

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4}$$

The role of γ_5 is played by $\sigma_3 = \gamma^0\gamma^1$. Consider a theory of N such fermions with Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i i\partial\!\!\!/ \psi - \frac{g^2}{2}\left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i\right)^2 . \tag{5}$$

As usual $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is the Dirac conjugate of the field ψ.

Find the dimensionality of the field ψ. Show that this theory is renormalizable in $1+1$ dimensions. Verify that γ_5 anticommutes with the other γ matrices. Show that the theory possesses a symmetry, $\psi_i \to \gamma_5\psi_i$, which forbids the generation of a mass term under renormalization. Then find the 1-loop beta function for the coupling g in this model. You should remember that a closed fermion loop gives an overall minus sign, and traces of single gamma matrices vanish. This will be useful to simplify some of the fermion loop diagrams in the computation.

Hint: you may find it useful to rewrite the interaction term in the path integral as

$$\exp\left(i\int d^2x\, \frac{-g^2}{2}\left(\sum_i \bar{\psi}_i\psi_i\right)^2\right) = \int \mathcal{D}\varphi\, \exp\left[i\int d^2x\, \left(\frac{1}{2g^2}\varphi^2 - \varphi\sum_i \bar{\psi}_i\psi_i\right)\right] . \tag{6}$$

Note that φ does *not* have a kinetic term; it is an auxiliary field whose Gaussian integral will generate the desired interaction term. What is the free propagator for the field φ?

3 Sine-Gordon Theory

Consider a $1+1$ dimensional theory with scalar fields $\phi(x,t)$ interacting with each other via the following Lagrangian:

$$\mathcal{L} = \int dx \left[\frac{1}{2}(\nabla_0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right] \tag{7}$$

where $U(\phi)$ is any function of ϕ bounded from below. Find the dimensionality of ϕ and argue that the theory remains renormalizable for *arbitrary* $U(\phi)$, that is, it does not have to be a polynomial of some maximal rank.

If $\phi_0(x)$ is a classical solution of the system (similar to the one that we discussed in the class), then show that the small fluctuations over this classical solution defined as:

$$\eta(x,t) \equiv \phi(x,t) - \phi_0(x) \equiv \sum_i c_i(t)\, \eta_i(x) \tag{8}$$

behave as an infinite number of harmonic oscillators with the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_i \left[\dot{c}_i^2 - \frac{1}{2}c_i^2\omega_i^2\right] + \text{constant} \tag{9}$$

where the spring constants ω_i^2 are the eigenvalues of a hypothetical Schrödinger equation with eigenfunction $\eta_i(x)$ given by:

$$\left[-\nabla^2 \;+\; \left(\frac{d^2U}{d\phi^2}\right)_{\phi_0(x)}\right]\eta_i(x) \;=\; \omega_i^2\;\eta_i(x)\,. \tag{10}$$

The above procedure works for *any* field theory that has a bounded potential. What can you say about the constant term in Eq. (9)?

A particularly interesting case is a potential of form

$$U(\phi) \;=\; \frac{m^4}{\lambda}\left[1 \;-\; \cos\left(\frac{\sqrt{\lambda}\phi}{m}\right)\right]. \tag{11}$$

The theory with this potential is called *Sine-Gordon theory.* Find the trivial vacua and the corresponding tower of states over these vacua. Sketch, but do not find the functional form of, one nontrivial vacuum. (That is, show what $\phi(x)$ should look like.) How many nontrivial vacua are there for this theory?

4 Renormalization of Yukawa theory

Consider again the Yukawa theory of the previous assignment, with massless scalars and supplemented by a scalar self-interaction:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4 - ig\bar{\psi}\gamma^5\psi\phi\,. \tag{12}$$

Compute the one-loop Beta functions for the couplings λ and g. (Use any results from last assignment–do not re-derive them for this assignment.)

5 Gravity

The Einstein-Hilbert action is

$$S_{\text{EH}} \;=\; \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R\,, \tag{13}$$

where G is the Newton's constant and R is the Ricci scalar.

Write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and consider h as the dynamical field. What is its dimension? [This is tricky–you might have to think.] Argue from this that the theory is non-renormalizable.

Suppose instead that, for some reason, the action were

$$S_{\text{modified}} \;=\; \int \sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)\,, \tag{14}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor (*i.e.* the traceless part of the Riemann tensor). Argue that this theory *is* renormalizable. Further, show that it possesses an invariance under the symmetry transformation $x \to \xi x$ with g fixed (or equivalently, $g_{\mu\nu} \to \xi^2 g_{\mu\nu}$ with x fixed). Do you expect this symmetry to survive renormalization? In the Wilsonian picture of renormalization, what other interaction term or terms might be generated in the RG flow of this theory towards the infrared?