Physics 673 Homework 3

Due 15 February 2007

1 Field strength

Show explicitly from the transformation properties of A^μ,

$$A_a^\mu \;\to_\omega\; A_a^\mu + \partial^\mu \omega_a - g f_{abc} \omega_b A_c^\mu \,, \tag{1}$$

and the definition of the covariant derivative $D^\mu = \partial^\mu - igA_a^\mu T^a$ that the field strength

$$\frac{i}{g}\Big[D_\mu \,,\, D_\nu\Big] \equiv F_{\mu\nu}^a T^a \tag{2}$$

transforms as an adjoint-valued field, independent of the representation for the covariant derivative (that is, for any representation of T^a provided that it satisfies the Lie algebra of the group). Then show that $F_{\mu\nu}^a F_a^{\mu\nu}$ transforms as a singlet (is gauge invariant). [Use your first result to do this–if you do, it is easy. If you go back to the transformation rule for A, it is a pain.]

2 SU(3)

The standard choice and normalization for the generators of the group SU(3) are $\frac{1}{2}$ times the *Gell-Mann matrices*, given in the appendix. Compute the structure functions for these matrices,

$$\left[\frac{\lambda_a}{2}\,,\,\frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}\,. \tag{3}$$

Note that, although there are 512 terms, most are zero; only write the nonzero ones not determined by antisymmetry in the first 2 entries. Is f_{abc} antisymmetric on *all* indices? [There may be ways to reduce the number of calculations you need to do but this is basically a “brute force” problem. Still, if you put your head down and do it, it should take less than an hour.]

Verify from the Gell-Mann matrices that $C = \frac{1}{2}$ and $C_2 = \frac{4}{3}$ for the fundamental representation (the representation the Gell-Mann matrices generate). Verify from your explicit results for f_{abc} that $C = 3$ and $C_2 = 3$ for the adjoint representation.

3 SU(5)

What are the dimensions of the symmetric and antisymmetric tensor representations of SU(5),

$$\mathbf{5} \otimes \mathbf{5} = ? \oplus ? \tag{4}$$

4 Lattice gauge theory

Lattice gauge theory is a regularization scheme, typically for Euclidean space quantum field theory, in which spacetime is replaced by a discrete lattice of points,

$$x^\mu = a n^\mu \,, \qquad n^\mu = (n^0, n^1, n^2, n^3) \,, \quad n^\mu \in \mathcal{Z} \,. \tag{5}$$

That is, n^μ is a quadruplet of integers. Here a is the lattice spacing, which serves as the UV regulator and should be smaller than any length scale of interest. Fields ϕ, ψ are restricted to "live" at the lattice points.

4.1 Discrete gradient terms

Show that, if a scalar field ϕ varies more slowly in space than the scale a, that

$$\mathcal{L}_{\rm latt} = \frac{a^2}{2} \sum_{x,\mu} \Big(\phi(x + a\hat{\mu}) - \phi(x) \Big)^\dagger \Big(\phi(x + a\hat{\mu}) - \phi(x) \Big) \tag{6}$$

is a good approximation for $\mathcal{L} = \int d^4x \frac{1}{2} (\partial_\mu \phi)^\dagger (\partial^\mu \phi)$. [There is no minus sign for the space components because we are in Euclidean space, $\eta_{\mu\nu} = \delta_{\mu\nu}$.]

Find a similar lattice approximation for $\bar{\psi}\gamma^\mu \partial_\mu \psi$. BE CAREFUL to make sure that your choice respects those rotations and reflections which carry the lattice to itself; otherwise the lattice theory would not recover rotational symmetry even in the small a limit.

4.2 Lattice gauge theory–first try

Consider an SU(2) Yang-Mills theory with a fundamental representation scalar field φ. Assume both $\varphi(x)$ and $A_\mu(x)$ are defined only at the lattice points.

Write a "naive" discretization for $(D_\mu \varphi)^\dagger (D^\mu \varphi)$. Show that it is *not* invariant under gauge transformations

$$\varphi(x) \;\rightarrow\; V(x)\varphi(x) \,, \qquad A_\mu(x) \;\rightarrow\; V(x) A_\mu(x) V^\dagger(x) - V(x) \partial_\mu V^\dagger(x) \tag{7}$$

for the simplest lattice meaning for $V^\dagger \partial_\mu V(x)$. [Actually this is true for *any* lattice version of the transformation law.]

4.3 Lattice gauge theory–right way

The geometrical meaning of the gauge field is as a *connection* which tells how to compare fields such as φ at different points. With this in mind, replace the gauge field $A_\mu(x)$ of the previous subproblem with the Wilson line connecting neighboring points on the lattice, $U(x+a\hat{\mu},x) \in \mathrm{SU}(2)$, defined on the "links" of the lattice (elementary lines between nearest neighbors). Gauge transformations are defined as

$$\varphi(x) \;\to\; V(x)\varphi(x)\,, \qquad U(x+a\hat{\mu},x) \;\to\; V(x+a\hat{\mu})U(x+a\hat{\mu},x)V^\dagger(x)\,. \tag{8}$$

Show that

$$\mathcal{L} = \frac{a^2}{2}\sum_{x,\mu}\left[\varphi(x+\hat{\mu}) - U(x+a\hat{\mu},x)\varphi(x)\right]^\dagger\left[\varphi(x+\hat{\mu}) - U(x+a\hat{\mu},x)\varphi(x)\right] \tag{9}$$

is invariant under gauge transformations, so it is a viable choice for the scalar lattice kinetic term.

Define $U(x,x+a\hat{\mu}) \equiv U^\dagger(x+a\hat{\mu},x)$. Show that $U(x,x+a\hat{\mu}) \to V(x)U(x,x+a\hat{\mu})V^\dagger(x+a\hat{\mu})$ under gauge transformations. Show that

$$P_{\mu\nu}(x) \equiv U(x,x+a\hat{\nu})U(x+a\hat{\nu},x+a\hat{\mu}+a\hat{\nu})U(x+a\hat{\nu}+a\hat{\mu},x+a\hat{\mu})U(x+a\hat{\mu},x) \tag{10}$$

transforms as an element of the adjoint representation at the point x, $P_{\mu\nu}(x) \in \mathrm{SU}(2) \to V(x)P_{\mu\nu}(x)V^\dagger(x)$.

For small and slowly varying gauge fields,

$$U(x+a\hat{\mu},x) \simeq 1 - iaA^a_\mu(x+a\hat{\mu}/2)T^a - \frac{a^2}{4}A^a_\mu A^a_\mu + O(A^3)\,, \tag{11}$$

show that $P_{\mu\nu}(x) \simeq 1 - ia^2 F^a_{\mu\nu}T^a$ (that is, verify that both the field derivative and commutator terms, but nothing else, arise at $O(a^2)$ when you expand out $P_{\mu\nu}(x)$ in terms of A as above and take A to be slowly varying). Then show that $\mathrm{Tr}\,P_{\mu\nu}(x)$ is gauge invariant, and argue from the approximate form for $P_{\mu\nu}(x)$ in terms of the field strength and from the fact that $P_{\mu\nu}(x) \in \mathrm{SU}(2)$ that

$$\mathrm{Tr}\,P_{\mu\nu}(x) \simeq 2 - \frac{a^4}{4}F^a_{\mu\nu}F^a_{\mu\nu} \tag{12}$$

(no sum on μ,ν implied). Therefore the sum over μ,ν squares of such traces could serve as a lattice version of the field strength.

A Gell-Mann matrices

The explicit form of the Gell-Mann matrices is

$$
\begin{aligned}
\lambda_1 &\equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad & \lambda_2 &\equiv \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\lambda_3 &\equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad & \lambda_4 &\equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
\lambda_5 &\equiv \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad & \lambda_6 &\equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
\lambda_7 &\equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad & \lambda_8 &\equiv \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\end{aligned}
\tag{13}
$$