Physics 673 Homework 4

Due 6 March 2007

1 Simple scattering color factors

Compute the squared matrix element, averaging over initial and summing over final "color" states, for the scattering of dissimilar, fundamental representation fermions in SU(N) gauge theory, $\psi_1\psi_2 \to \psi_1\psi_2$, at leading order.

The diagram is the same as for $e^-\mu^- \to e^-\mu^-$ scattering, and so is the whole calculation except the part involving group theoretical ("color") factors. You should be able to find an expresion only in terms of group and representation invariants, eg, $C_2[R]$, $C[R]$, d_R, *etc.*

2 Quark production

[This problem is easy once you understand what it is asking.]

Consider the theory of QED plus QCD: the gauge group is SU(3)×U(1), where U(1) is conventional electromagnetism and SU(3) is the "color" nonabelian gauge group. There are the following Dirac fermions: e and μ, each charge -1 and a singlet under SU(3); d and s, each charge -1/3 and a triplet (fundamental) under SU(3); and u, charge 2/3 and a triplet (fundamental) under SU(3).

Consider the reaction $e^+e^- \to \bar{\psi}\psi$, with ψ any of μ, d, s, u. All processes proceed via an intermediate s-channel photon. Draw the diagram, write down the squared matrix element for the case $\psi = \mu$ (just use results from last term), and compute the squared matrix element, summed over colors, for each of the states d, s, u at leading order. Assume that the masses of these particles are small compared to the energy available, that is, $s \gg m_d^2, m_s^2, m_u^2$. [Here s is the Mandelstam variable, not the field s.]

Basically all you need to do this problem is to compute the "color factor." How is the calculation different (simpler!) than the one in the previous problem?

What is the ratio of quark pair production to muon pair production? This should be a pure number.

3 Beta function with scalars

Consider adding a complex scalar field in the fundamental representation to SU(N) Yang-Mills theory. The scalar φ can be thought of as an N component column vector of complex numbers, $\varphi = [(\phi_{1r}+i\phi_{1i})/\sqrt{2}\,,\ \phi_{2r}+i\phi_{2i})/\sqrt{2}\,,\ \ldots]^\top$. (It is conventional to write the complex components $\varphi_1, \varphi_2, \ldots$ in terms of real components with this square root of two, so that $d\phi_{1r} d\phi_{1i} = d\varphi_1 d\varphi_1^*$.)

The new terms which can appear in the Lagrangian are

$$\mathcal{L}_\varphi = (D_\mu \varphi)^\dagger (D^\mu \varphi) - m^2 \varphi^\dagger \varphi - \frac{\lambda}{2} (\varphi^\dagger \varphi)^2 \,, \tag{1}$$

where the covariant derivative is $D_\mu = \partial_\mu - ig A_\mu^a T^a$ with T^a in the fundamental representation.

Write the Feynman rules for the scalar propagator, the $\varphi^\dagger \varphi A$ vertex, the $\varphi^\dagger \varphi AA$ vertex, and the $\varphi^\dagger \varphi \varphi^\dagger \varphi$ vertex.

Write out the expressions for the two new A-field self-energy diagrams arising from scalar loops. (One involves the $\varphi^\dagger \varphi AA$ vertex, the other involves two $\varphi^\dagger \varphi A$ vertices.) Evaluate the scalar's contribution to the one-loop gauge coupling beta function. Do this by finding the $O(q^2)$ coefficient on the $1/\epsilon$ part of the diagrams in $\overline{\mathrm{MS}}$ renormalization, or finding the log divergent part after double differentiating with respect to q. If you use $\overline{\mathrm{MS}}$, you should also verify that there is no pole at $d = 2$ (no gauge boson mass induced by scalar loops).

What is the sign of the correction from a scalar field? Does it make the theory more or less asymptotically free?

Extra credit: compute the 1-loop beta function for the gauge coupling in $\mathcal{N}$=4 supersymmetric Yang-Mills theory, which is a theory with 3 complex *adjoint* representation scalar fields and 4 *adjoint* representation Weyl fermions, equivalent to 2 adjoint representation Dirac fermions.

Extra extra credit: compute the 1-loop beta function for the scalar coupling λ. This involves finding the $O(q^2)$ divergent part of the scalar field wave-function renormalization and the zero-momentum divergent part of the scalar 4-point function. You will find it easiest to proceed in Landau gauge, where all the 4-point diagrams with $\varphi^\dagger \varphi A$ vertices give zero.

4 Ghost 4-point interaction?

Argue that the scaling dimension of the ghost field c is 1, like other boson fields. Therefore, a diagram with two c and two $\bar{c}$ external lines is potentially logarithmically divergent, and we could imagine that such an interaction is generated by renormalization and must be included in the Lagrangian.

Let us see why this does not happen. Draw a 1-loop diagram with two incoming and two outgoing ghost lines, and use power counting estimates on each vertex and propagator to show that its superficial degree of divergence is 0 (log divergent). Now write down an expression for the actual contribution of this diagram. You should find that it is in fact UV convergent by two powers. How did that happen? Find a simple argument to show that this will occur at all loop orders, so in fact no $\bar{c}c\bar{c}c$ vertex is generated under renormalization.