Physics 673   Homework 5

Due 22 March 2007

1 Chiral anomaly

Consider QED + QCD, with 3 flavors of quarks, u, d, s, with electric charges 2/3, −1/3, and −1/3;

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \sum_{f=uds} \bar{q}_f(i\not{D} - m_f)q_f \,, \tag{1}$$

with $D_\mu = \partial_\mu - igT^aG^a_\mu + iQ_f eA_\mu$ and $Q_u = 2/3$, $Q_d = Q_s = -1/3$. Here A and G are the electromagnetic and color gauge fields and $F_{\mu\nu}, G_{\mu\nu}$ are the respective field strengths.

First, show that the electromagnetic and color currents are anomaly free. (This is easy–remember that right and left handed components of fields contribute oppositely to the anomalous current.)

Consider the symmetry transformation $q_u \to e^{i\theta\gamma_5}q_u$, with q_u the up quark field. Show that this transformation is *not* a symmetry of the Lagrangian, but would be a symmetry if the up quark were massless.

Now find the divergence in the current associated with this symmetry,

$$\partial_\mu J^\mu_{u5} \equiv \partial_\mu \bar{q}_u\gamma^\mu\gamma^5 q_u \,, \tag{2}$$

for the case of a massless and a massive up quark.

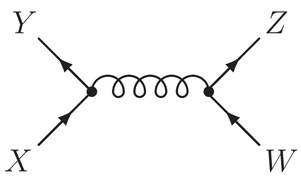
Hint: for the massless case, there are two contributions arising because of the chiral anomaly, one arising from the electromagnetic interactions and proportional to e^2 and one arising from strong interactions and proportional to g^2. Be very careful finding the overall coefficients. Why is the other possible term, proportional to eg, absent?

2 Color Charges

The colors of the 8 gluons can be represented as follows:

$$\begin{array}{lll} g_1 = R\bar{G} & g_2 = R\bar{B} & g_4 = G\bar{R} \\ g_5 = G\bar{B} & g_6 = B\bar{R} & g_7 = B\bar{G} \\ g_3 = \frac{1}{\sqrt{2}}(R\bar{R} - G\bar{G}) & g_8 = \frac{1}{\sqrt{6}}(R\bar{R} + G\bar{G} - 2B\bar{B}) & \end{array} \tag{3}$$

The numerical values associated with the ‘colors’ can be now read off from these.



Consider the above diagram. Time runs in the upward direction. The colors W, X, Y, Z are not arbitrary. Once (X, Y) are fixed, then the glue that is exchanged must include $X\bar{Y}$. This then limits (W, Z) to be either (X, Y) or $(\bar{X}, \bar{Y})$. Consider now the case $(X, Y) = (B, B)$ and $(W, Z) = (B, B)$. Then the charge factors associated with this case are as follows. The incoming particles are both quarks. Therefore there are two factors of $+g$'s. (When incoming particle is an antiquark, that vertex gets a $-g$.) The only gluon that contains $B\bar{B}$ is g_8. The coefficient that is associated with $B\bar{B}$ component is $-2/\sqrt{6}$. Since there are two vertices where the gluon interacts, there are two factors of $(-2/\sqrt{6})$. All combined, this gives

$$C_{B\bar{B}} = g^2 \left(\frac{-2}{\sqrt{6}} \right)^2 = \frac{2g^2}{3} \tag{4}$$

2.1

Verify that $RR \to RR$ and $GG \to GG$ gets the same weight as the $BB \to BB$ case. Remember that both g_3 and g_8 come into play now.

2.2

Verify that the weight associated with $q\bar{q} \to q\bar{q}$ (all same color) is $-2g^2/3$.

2.3

Verify that the weight associated with $q\bar{q} \to q'\bar{q}'$ is $-4g^2/3$ when the color of the $q'\bar{q}'$ pair is summed over. Consider only the process that's shown in the above figure. Remember that there is a conventional overall factor of 1/2.

2.4

Can B and $\bar{R}$ interact by exchanging a single gluon? Explain.

3 Hadronic Tensor

3.1 QED + QCD

When one needs to consider photons as well as gluons, the covariant derivative becomes

$$D_\mu = \partial_\mu - ig\, t_a\, A^a_\mu + ieQ_q\, A_\mu \tag{5}$$

where A^a_μ is the gluon field and A_μ is the photon field. The Lagrangian is given by

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + i\sum_f \bar{q}_f \not{D} q_f \tag{6}$$

where $F_{\mu\nu}$ is the EM field strength tensor and $G^a_{\mu\nu}$ is the QCD field stregth tensor and we have set all quark masses to 0.

3.2 Equations of Motion

By taking appropriate functional derivatives, derive the following equations of motion for A_μ, A^a_μ and q_f.

$$\not{D} q_f \;=\; 0 \tag{7}$$

$$\partial_\mu F^{\mu\nu} \;=\; e\sum_f Q_f \bar{q}_f \gamma^\nu q_f = J^\nu_{\rm EM} \tag{8}$$

$$\left(D_\mu G^{\mu\nu}\right)_a \;=\; -g\sum_f \bar{q}_f \gamma^\nu t_a q_f = J^\nu_a \tag{9}$$

where the covariant derivative appearing in the last line is the adjoint covariant derivative and contains only the gluon field. Work out carefully what we mean by $(D_\mu G^{\mu\nu})_a$ here.

Show that the electromagnetic current $J^\nu_{\rm EM}$ satisfies

$$\partial_\nu J^\nu_{\rm EM} = 0 \tag{10}$$

and the color current J^ν_a satisfies

$$(D_\mu J^\mu)_a = 0 \tag{11}$$

Does this equation mean that a color charge is conserved? Explain your answer.

3.3 Hadronic tensor in $e^+e^- \to \gamma^* \to q\bar{q} + X$

We have seen that in both the $e^+e^- \to q\bar{q}$ process and the $e^+e^- \to q\bar{q}g$ process, the scattering matrix element squared can be decomposed into the lepton part $L_{\mu\nu}$ and the hadronic (quark) part $H^{\mu\nu}$. Rewrite the interaction terms in the QCD + QED Lagrangian using the currents above and show that the hadronic tensor $H^{\mu\nu}$ must be transverse, that is, $\partial_\mu H^{\mu\nu} = 0$ or in momentum space $q_\mu H^{\mu\nu} = 0$ where q is the γ^* mometum.

Suppose we consider instead the $q\bar{q} \to g^* \to q\bar{q}$ process. Are the tensors involved in this matrix element squared still regarded as transverse? Explain.

4 2 to 2 scatterings in QCD

4.1 $q\bar{q} \to gg$

1. Draw all tree level diagrams corresponding to the $q\bar{q} \to gg$ process.

2. Square the sum of the resulting matrix elements, average over initial states, sum over final states and integrate over the appropriate 2 particle phase space to yield

$$\frac{d\sigma}{dt}(q\bar{q} \to gg) = \frac{32\pi\,\alpha^2}{27\,s^2}\left(\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\left(\frac{t^2+u^2}{s^2}\right)\right) \tag{12}$$

   [This is the part where you have to be careful either to use only transverse polarization states for the gluons or to include ghosts in the final state sum.]

   Since one of the terms enters with a minus sign, one must check that this result is positive for all physical values of s, t, u. Is this so? [Take all external masses to be zero.]

3. Use crossing to deduce that

$$\frac{d\sigma}{dt}(gg \to q\bar{q}) = \frac{\pi\,\alpha^2}{6\,s^2}\left(\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\left(\frac{t^2+u^2}{s^2}\right)\right) \tag{13}$$

4.2 $qg \to qg$

Draw all tree level diagrams for the $qg \to qg$ process. Deduce from crossing that

$$\frac{d\sigma}{dt}(gg \to q\bar{q}) = \frac{4\pi\,\alpha^2}{9\,s^2}\left(-\frac{u}{s} - \frac{s}{u} + \frac{9}{4}\left(\frac{s^2+u^2}{t^2}\right)\right) \tag{14}$$

Is this positive? How did the signs turn out the way they did?