Appendix B
Symmetries and group theory review

We give here a very short review of symmetries, and of the theory of Lie groups needed in their study. The treatment is not intended to be complete or rigorous, just to give a brief introduction for the readers unfamiliar with the material.

B.1 Symmetry transformations as a group

A symmetry transformation is a transformation on the states of a theory $|\psi\rangle$ and the operators $\mathcal{O}$,

$$|\psi\rangle \;\rightarrow\; U|\psi\rangle\,, \qquad \mathcal{O} \;\rightarrow\; U\,\mathcal{O}\,U^*\,, \tag{B.1}$$

which preserves "all physics." In particular, amplitudes must be preserved (up to a phase, a complication we ignore here and in the following)

$$\langle\psi_1|\,|\psi_2\rangle \;\rightarrow\; \langle\psi_1|\,U^*\,U\,|\psi_2\rangle = \langle\psi_1|\,|\psi_2\rangle\,, \tag{B.2}$$

which shows that U must either be unitary or anti-unitary. We will only consider unitary symmetries here. (Here U^* is the Hermitian conjugate of U. We write $U^\dagger$ only if U is a matrix.)

A symmetry in which a local operator $\mathcal{O}(x)$ (an operator built out of fields at point x only) is transformed into a local operator at the same point, is called an *internal symmetry*. Such symmetries can be considered separately from *spacetime symmetries*, such as translations, rotations, and boosts. In fact, it is a theorem that the full group of symmetries is always a product of the internal symmetries and the spacetime symmetries. In this appendix we concentrate on internal symmetries; spacetime symmetries are discussed in the next appendix.

Because the states and operators under discussion may appear at different times, the symmetry operator must commute with time evolution,

$[H, U] = 0$. Similarly, it must commute with the momentum operators. This is summarized by saying that it must commute with the Lagrangian density, $[\mathcal{L}, U] = 0$. Therefore the symmetries of the theory can usually be identified by looking at the symmetries of the Lagrangian.

In a renormalizable theory, an operator $\mathcal{O}$ and its symmetry transform $U\mathcal{O}U^* = \mathcal{O}'$ must be of the same dimension. Since the fields are the operators of the smallest dimension, this means the fields transform linearly among themselves,

$$\varphi_a \;\rightarrow\; U\varphi_a U^* = M^{-1}_{ab}\varphi_b\,, \tag{B.3}$$

where φ collectively symbolizes the fields of the theory and a, b are indices on the set of fields. Classically, the symmetries of a theory are the set of transformations on the fields of this form, under which $\mathcal{L}$ is unchanged. The relation between φ' and φ involves a matrix inverse essentially because φ annihilates a particle $|\varphi\rangle$, and must therefore have the inverse transformation properties of the particle.

Acting with two symmetry transformations successively, $|\psi\rangle \rightarrow U_1|\psi\rangle \rightarrow U_2U_1|\psi\rangle$, yields another symmetry transformation, namely, the one induced by the operator (U_2U_1). This defines a multiplication rule under which symmetry transformations form a group. In general the symmetry group can be factorized as a product of non-factorizable subgroups, and it is sufficient to examine the behavior of the subgroups individually. In particle physics these subgroups are usually small discrete groups (which we will not discuss) and continuous (Lie) groups. The latter can generally be factorized into a product of simple Lie groups and $U(1)$ groups.

B.2 Lie groups and Lie algebras

A Lie group is a group which is also a manifold. In particular, there is a small neighborhood around the identity $\mathbf{1}$ which looks like a piece of $\mathrm{Re}^{\,n}$, with n the dimension of the group. One can always choose a coordinate basis for this region; the coordinate unit vectors t_a are called the *Lie algebra* and an arbitrary element g of the group which is close to the identity can always be expanded in the coordinates,

$$g = \mathbf{1} + i\omega^\alpha t_\alpha\,, \tag{B.4}$$

with ω_α (infinitesimal) parameters. (The i is customary so that for groups of unitary matrices, the t_α are Hermitian.)

Now consider two elements of the group which are each close to the identity, say, $g_1 = \mathbf{1} + i\omega_1^\alpha t_\alpha$ and $g_2 = \mathbf{1} + i\omega_2^\alpha t_\alpha$. The multiplication rule to first order in these parameters is given by

$$g_1 g_2 = \mathbf{1} + i(\omega_1^\alpha + \omega_2^\alpha)t_\alpha + O(\omega^2)\,, \tag{B.5}$$

which is addition of the departures from the identity. At the next order, $g_1 g_2$ and $g_2 g_1$ can differ:

$$\begin{aligned} g_1 g_2 (g_2 g_1)^{-1} &= (\mathbf{1} + i\omega_1^\alpha t_\alpha)\left(\mathbf{1} + i\omega_2^\beta t_\beta\right)(\mathbf{1} - i\omega_1^\gamma t_\gamma)\,(\mathbf{1} - i\omega_2^\sigma t_\sigma) \\ &= \mathbf{1} - \omega_1^\alpha \omega_2^\beta \Big[t_\alpha\,,\, t_\beta\Big]\,. \end{aligned} \tag{B.6}$$

Therefore, to determine the multiplication rule to second order we need to know the commutators of the Lie algebra elements. Since $g_1 g_2 g_1^{-1} g_2^{-1}$ is still close to the identity, it can still be expressed in terms of coefficients multiplying Lie algebra elements, so the commutator must also be a sum of elements of the Lie algebra:

$$\Big[t_\alpha\,,\, t_\beta\Big] = i f^\gamma{}_{\alpha\beta} t_\gamma\,. \tag{B.7}$$

The *structure constants*, $f^\gamma{}_{\alpha\beta}$ are real valued and explicitly anti-symmetric in the last two indices, and are antisymmetric in all indices if the t_α are chosen orthonormal.

The Lie algebra elements and the structure constants together constitute the *Lie algebra* of the group. They turn out to be sufficient to determine the group almost uniquely.†

The groups of interest in particle physics are compact Lie groups. These can all be thought of as groups of matrices. Of particular interest is the group of $N \times N$ special (unit determinant) unitary matrices, $SU(N)$, which we describe in more detail in the next section.

B.3 Group representations

We saw in Eq. (B.3) that a symmetry transformation acts on a field operator like a matrix multiplication. Successive symmetry transformations act like a series of matrix multiplications, which gives us a condition on the matrices which can appear in Eq. (B.3). Namely, under successive transformations by two group elements,

$$U(g_2 g_1)\varphi_a U^*(g_2 g_1) \quad = \quad U(g_2)U(g_1)\varphi_a U^*(g_1)U^*(g_2)$$

† A Lie group can have several disconnected pieces; the Lie algebra specifies only the connected piece containing the identity. For simple compact Lie groups, the Lie algebra gives a unique simply connected group, and any other connected group with the same Lie algebra must be must be a quotient of this group over a discrete normal subgroup. For instance, the Lie algebra of rotations gives the group $SU(2)$. The group $SO(3)$ has the same Lie algebra, but differs in that a rotation by 360°, represented in $SU(2)$ by Diag$[-1\ \ -1]$, is identified with the identity in $SO(3)$.

$$\begin{aligned} M_{ab}^{-1}(g_2 g_1)\varphi_b &= U(g_2) M_{ab}^{-1}(g_1)\varphi_b U^*(g_2) \\ M_{ac}^{-1}(g_2 g_1)\varphi_c &= M_{ab}^{-1}(g_1) M_{bc}^{-1}(g_2)\varphi_c \,. \end{aligned} \tag{B.8}$$

Since this must hold for any field φ, the matrices themselves must be equal,

$$M^{-1}(g_2 g_1) = M^{-1}(g_1) M^{-1}(g_2) \quad \text{or} \quad M(g_2 g_1) = M(g_2) M(g_1)\,. \tag{B.9}$$

Matrix multiplication must respect group element multiplication. A set of matrices associated with elements of a group which satisfy this condition are called a *representation* of the group. Since the matrices can be thought of as operating on column vectors, physicists often refer to column vectors (or fields) which are multiplied by such matrices as representations. More properly, one should say that such column vectors or fields are "acted on" or "transform under" the representation. To understand the ways symmetries can act on fields and field products we must understand representations and their tensor products.

In any representation, the identity element of the group must be mapped into the identity matrix $\mathbf{1}$. An element close to the identity must map into an element close to the identity matrix, so

$$M(\mathbf{1} + i\omega^\alpha t_\alpha) = \mathbf{1} + i\omega^\alpha T_\alpha \,, \tag{B.10}$$

with T_α some matrices particular to the representation. (We use Greek letters to index the Lie algebra and Roman letters for matrix indices.) It then follows by considering products of such matrices and using Eq. (B.9) that the matrices T_α must satisfy a Lie algebra with the same structure constants as the t_α:

$$\Big[T_\alpha \,,\, T_\beta \Big] = i f^\gamma{}_{\alpha\beta} T_\gamma \,. \tag{B.11}$$

Furthermore, any set of matrices T_α which satisfy this identity can be exponentiated to give a representation. Frequently a basis of fields can be found under which the T_α are all block diagonal, in which case the representation is said to *reduce* into the blocks. A representation wich cannot be block diagonalized by any basis change is called *irreducible*. The problem of classifying representations of a group G is the problem of finding all sets of matrices T_α which obey the same commutation relations as the Lie algebra of the group.

Every group has a representation, called the *singlet* or *trivial* representation, in which $M(g)$ is the 1×1 identity matrix for each g, and $T_\alpha = 0$ for all α. (Eq. (B.9) is satisfied because $1 \times 1 = 1$, and Eq. (B.11) is automatically satisfied since both sides are zero.) Invariance of the Lagrangian under a symmetry is equivalent to the requirement that the Lagrangian transform under the singlet representation. Therefore it will be important to see how other representations can be combined together to give the singlet representation.

Every group also contains a representation called the *adjoint* representation, made up of $n \times n$ real matrices, with n the number of elements in the Lie algebra, and with T_α given by $(T_\alpha)_{bc} = -if^\alpha{}_{bc}$ (with b, c the matrix indices). For the case of an abelian group (a group where the f vanish) the adjoint representation is the same as the singlet representation. For the group of rotations, $SU(2)$, it is the spin-one representation.

Physicists generally refer to the several fields which transform together in an irreducible representation of the symmetry group as "a" field transforming under that representation. If two such fields ϕ, χ transform under two different representations with representation matrices M, N which are respectively $m \times m$ and $n \times n$, then the operator $\phi_a \chi_b$ transforms as

$$U(g^{-1})\phi_a \chi_b U^*(g) = M_{ac}(g) N_{bd}(g)\, \phi_c\, \chi_d\,. \tag{B.12}$$

The object $M_{ac}N_{bd}$ can be considered an $(mn) \times (mn)$ matrix obtained as the tensor product of the matrices M and N. So the product of two operators transforms under the tensor product of the representation matrices. In general such tensor products are reducible–for instance, in the familiar example of angular momentum $SU(2)$, two spin-half operators can combine into a spin-one or a spin-zero operator, because the tensor product $\frac{1}{2} \otimes \frac{1}{2}$ is reducible, $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

Representations and the rules for their tensor products are quite group dependent. We will quickly outline what happens for $U(1)$ and $SU(N)$, since these arise the most often in physics and in particular are the only groups needed in the standard model.

B.3.1 Representations of $U(1)$

The group $U(1)$ is the group of phase rotations. A generic element is $e^{i\theta}$ and the group is parametrized by θ. Any irreducible representation can be written as a 1×1 complex matrix (a complex number) and the representation is determined by a charge q, with the group element $g = e^{i\theta}$ being represented by $e^{iq\theta}$. The tensor product of two representations is just a representation with the sum of the charges. Therefore, the charge of a product of operators is the sum of their charges. For the Lagrangian to have a $U(1)$ symmetry, each term in the Lagrangian must have the charges of the fields add up to zero.

B.3.2 Representations of $SU(N)$

The group $SU(N)$ consists of complex $N \times N$ matrices U which are unitary, $U^\dagger = U^{-1}$, and satisfy $\operatorname{Det} U = 1$. (The U in $SU(N)$ stands for unitary, the S for "special," meaning determinant 1.)

A generic element of $SU(N)$ can be written $U(\omega) = \exp(i\omega_\alpha t_\alpha)$, with t_α a standard set of $N \times N$ complex matrices and ω_α parameters. Before imposing unitarity and determinant 1, there are $2N^2$ independent t_α. However, unitarity requires each t_α be Hermitian, eliminating half, and the unit determinant condition requires each t_α be traceless, eliminating one more possibility. Therefore there are $N^2 - 1$ independent elements t_α of the Lie algebra, which should be chosen to be orthogonal and to satisfy the same normalization condition. For $SU(2)$ they can be chosen to be half the Pauli matrices, Eq. (2.12). In this case the structure functions are

$$\left[\frac{\tau_i}{2}\,,\,\frac{\tau_j}{2}\right] = i f_{kij}\frac{\tau_k}{2}\,, \qquad f_{kij} = \epsilon_{kij} \tag{B.13}$$

the totally antisymmetric tensor. For $SU(3)$ the Lie algebra elements can be chosen to be half the Gell-Mann matrices of Eq. (1.186). There is no simple expression for the resulting structure functions.

Besides the singlet representation, the smallest representation for $SU(N)$ is the $SU(N)$ matrices themselves, $M(U) = U$. This is called the *fundamental representation*.

$$\text{Fundamental representation:}\quad T_\alpha = t_\alpha\,. \tag{B.14}$$

This is, for instance, the representation quarks transform under in QCD. It is customary to refer to the representations of $SU(N)$ according to the rank of the representation matrices, so the singlet representation is called the **1** representation and the fundamental representation is called the **N** representation.

Equally important is the *anti-fundamental representation*, $M(U) = U^*$, given by complex conjugating (but not transposing) the $SU(N)$ matrices.

$$\text{Anti-Fundamental representation:}\quad T_\alpha = -t_\alpha^*\,. \tag{B.15}$$

This is the representation which antiquarks transform under. To see that it is a valid representation, note that

$$\left[-t_\alpha^*\,,\,-t_\beta^*\right] = \left(\left[t_\alpha\,,\,t_\beta\right]\right)^* = (i f^\gamma{}_{\alpha\beta} t_\gamma)^* = i f^\gamma{}_{\alpha\beta}(-t_\gamma^*)\,. \tag{B.16}$$

Therefore the $-t_\alpha^*$ obey the same commutation relations as the t_α. These matrices have rank N, but since the symbol **N** is taken, the representation is called the $\overline{\mathbf{N}}$ representation.

A field which transforms under the fundamental representation is left-multiplied by U, and can be thought of as a column vector. For the anti-fundamental representation, the column vector is left-multiplied by U^*, and is more conveniently thought of as a row vector right-multiplied by $U^\dagger = U^{-1}$. In general, if M is a representation of a group, M^* is also, and is called the *conjugate representation* to M. The contraction of a fundamental and an anti-fundamental field (or any operators in conjugate representations to each other) forms a singlet,

$$\text{for } \chi_a \;\; \overline{\mathbf{N}}\,, \quad \phi_a \;\; \mathbf{N}\,, \qquad \chi^T \phi \equiv \chi_a \phi_a \text{ is singlet } \mathbf{1}\,. \tag{B.17}$$

At the same time, inserting the generators of the representation between them,

$$\text{for } \chi_a \;\; \overline{\mathbf{N}}\,, \quad \phi_a \;\; \mathbf{N}\,, \qquad \chi^T T_\alpha \phi \equiv \chi_a (T_\alpha)_{ab} \phi_b \;\text{ is adjoint}\,. \tag{B.18}$$

Since there are $N^2 - 1$ elements in the adjoint representation and one in the singlet, this uses up the $N \times N = N^2$ objects in the tensor product of fundamental and anti-fundamental representations;

$$\overline{\mathbf{N}} \otimes \mathbf{N} = (\mathbf{N^2} - \mathbf{1}) \oplus \mathbf{1}\,. \tag{B.19}$$

It is also important to know how multiple fundamental representations tensor together. Here it is important to know that the totally antisymmetric object $\epsilon_{ab\ldots}$, which contracts N fundamental indices, is an invariant. Using it to contract $N-1$ objects transforming in the fundamental representation gives an object transforming in the anti-fundamental representation. For $SU(2)$, this means that a single fundamental representation object can be "flipped" into an anti-fundamental representation object by ε, as we did with the Higgs field in Eq. (2.24). In $SU(2)$ the fundamental and anti-fundamental representations are equivalent and generally not distinguished from each other. In $SU(3)$, contracting two fundamental fields with the antisymmetric tensor, $\epsilon_{abc}\phi_b\psi_c$, produces an anti-fundamental object, and three gives a singlet. The other (symmetrized) linear combination of two fundamental fields has six components and is called the **6** representation:

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6} \qquad \text{in } SU(3)\,. \tag{B.20}$$

More generally one gets representations containing $N(N-1)/2$ and $N(N+1)/2$ elements. In general, $SU(N)$ groups have a large number of representations, all of which can be found by taking antisymmetrized and symmetrized combinations of fundamental representation objects. Further enumerating them is beyond the scope of this appendix.

Appendix C

Lorentz group and the Dirac algebra

This appendix provides a review and summary of the Lorentz group, its properties, and the properties of its infinitesimal generators. It then reviews representations of the Lorentz group and the Dirac algebra. This material is intended to supplement Chapter 1, for those students who are not as familiar with the Lorentz group and Dirac equation as they find they need to be.

C.1 Lorentz group

According to special relativity, physical laws are unchanged by a linear change of coordinates,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \xi^{\mu} \tag{C.1}$$

with Λ and ξ real, provided it leave unchanged the invariant separation between points,

$$(x-y)_{\mu}(x-y)^{\mu} = \eta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu} = -[(x-y)^{0}]^{2} + [\vec{x}-\vec{y}]^{2} \,.$$

This condition does not constrain ξ, since it cancels in the difference, but it imposes a constraint on Λ,

$$x_{\mu}x^{\mu} = x'_{\mu}x'^{\mu} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}x^{\alpha}\Lambda^{\nu}{}_{\beta}x^{\beta} \,, \tag{C.2}$$

for all x^{μ}. A transformation of the form shown in Eq. (C.1) which satisfies Eq. (C.2) is called a Poincaré transformation. These transformations close and form a group, called the Poincaré group. The subgroup where Λ is the identity matrix and ξ is arbitrary is a subgroup called the group of translations. We assume that this group and its implications, such as conservation of energy and momentum, are familiar to the reader. Instead we concentrate on the subgroup in which $\xi = 0$, which is called the Lorentz group.

It is convenient to think of an element of the Lorentz group as a matrix which is operating on the coordinate x^μ. This is possible if we always write Λ with its first index raised and second index lowered, so it carries x^μ to x'^μ, both with raised index. Writing in this way, repeated Lorentz transformations are implemented via matrix multiplications of the respective Λ matrices:

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \text{ and } x''^\mu = \Lambda'^\mu{}_\nu\, x'^\nu \;\Rightarrow\; x''^\mu = \Lambda'^\mu{}_\nu\, \Lambda^\nu{}_\alpha\, x^\alpha \equiv \left[\Lambda'\Lambda\right]^\mu{}_\alpha\, x^\alpha\,. \tag{C.3}$$

We see from Eq. (C.2) that the condition on $\Lambda^\mu{}_\nu$ to be a Lorentz transformation is,

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu x^\mu x^\nu \tag{C.4}$$

for all x^μ. Since this must hold for all x^μ, we have

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu\eta_{\alpha\beta}\Lambda^\beta{}_\nu\,, \tag{C.5}$$

or (writing $\eta_{\mu\nu}$ as η when using matrix notation)

$$\eta = \Lambda^T\eta\Lambda\,. \tag{C.6}$$

The group of matrices Λ satisfying Eq. (C.6) is called $O(3,1)$, and is a Lie group. Therefore the same technology of Lie algebra generation may be applied to it as to the groups of the previous appendix.

As we will discuss momentarily, not all elements of $O(3,1)$ can be built infinitesimally from the identity. Those elements which can, form a subgroup written $SO(3,1)$, which we will now analyze. A Lorentz transformation $\Lambda^\mu{}_\nu$ which is infinitesimally close to the identity must be of form,

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu\,, \tag{C.7}$$

with $\omega^\mu{}_\nu$ a matrix of infinitesimal coefficients. The condition on $\omega^\mu{}_\nu$ for $\Lambda^\mu{}_\nu$ to be a valid Lorentz transformation is found by inserting Eq. (C.7) into Eq. (C.6) and expanding to linear order in ω:

$$\begin{aligned} \eta_{\mu\nu} &= \left(\delta^\alpha_\mu + \omega^\alpha{}_\mu\right)\eta_{\alpha\beta}\left(\delta^\beta_\nu + \omega^\beta{}_\nu\right) \\ &= \eta_{\mu\nu} + (\omega_{\nu\mu} + \omega_{\mu\nu}) + O(\omega^2)\,, \\ 0 &= \omega_{\nu\mu} + \omega_{\mu\nu}\,. \end{aligned} \tag{C.8}$$

That is, the condition on $\omega^\mu{}_\nu$ is that $\omega_{\mu\nu}$ be antisymmetric on its indices. The space of real antisymmetric 4×4 matrices is 6 dimensional, so the Lorentz group is 6 dimensional.

Now $\omega^\mu{}_\nu$ is related to $\omega_{\mu\nu}$ as $\omega^\mu{}_\nu = \eta^{\mu\alpha}\omega_{\alpha\nu}$. Since $\eta^{00} = -1$ and $\eta^{ii} = 1$ for $i = 1,2,3$, the sign of the space-time component of $\omega^\mu{}_\nu$ must be the same as the sign of the time-space component, while the space-space components must be antisymmetric. Thus, the most general form of $\omega^\mu{}_\nu$ is,

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix}, \tag{C.9}$$

symmetric in the space-time entries and anti-symmetric in the space-space entries. The b_1, b_2, b_3 entries respectively cause infinitesimal boosts in the $1, 2, 3$ directions; the r_1, r_2, r_3 entries cause infinitesimal rotations about the $1, 2, 3$ axes. A general element of $SO(3,1)$ can be written as an exponential of a *finite* $\omega^\mu{}_\nu$,

$$\Lambda^\mu{}_\nu = (\exp \omega)^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \frac{1}{2}\omega^\mu{}_\alpha \omega^\alpha{}_\nu + \frac{1}{6}\omega^\mu{}_\alpha \omega^\alpha{}_\beta \omega^\beta{}_\nu + \ldots . \tag{C.10}$$

When only the r_i are nonzero, this gives a rotation by angle $|\vec{r}|$ about the $\hat{r}$ axis. When only the b_i are nonzero, this gives a boost by velocity $\tanh |\vec{b}|$ along the $\hat{b}$ axis. When both $\vec{r}$ and $\vec{b}$ are nonzero the Lorentz transformation cannot be described either solely as a rotation or as a boost. Note that, while a rotation by angle $|\vec{r}| = 2\pi$ gives the identity Λ, no nonzero magnitude of boost $|\vec{b}|$ returns the identity. Hence the group $SO(3,1)$ is *non-compact*.

Now we argue that $O(3,1)$ has 4 disconnected pieces, one of which is $SO(3,1)$. To see this, take the determinant of Eq. (C.6):

$$\text{Det}\,\eta = \text{Det}\,\Lambda^T \eta \Lambda = \text{Det}\,\eta \times (\text{Det}\,\Lambda)^2 . \tag{C.11}$$

Since η is nonsingular, we can divide by Det η:

$$(\text{Det}\,\Lambda)^2 = 1 \quad \Rightarrow \quad \text{Det}\,\Lambda = \pm 1 . \tag{C.12}$$

The determinant must vary continuously within a path connected region of $O(3,1)$, but you cannot go discontinuously from 1 to -1, so any elements of $O(3,1)$ with $\text{Det}\,\Lambda = -1$ cannot be elements of the connected group $SO(3,1)$. An element of $O(3,1)$ with $\text{Det}\,\Lambda = 1$ is called *proper*, and an element with $\text{Det}\,\Lambda = -1$ is called *improper*.

Furthermore, if we write out the $\mu = 0$, $\nu = 0$ element of Eq. (C.6), it is

$$\begin{aligned} \eta_{00} &= \Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 \\ -1 &= -\Lambda^0{}_0 \Lambda^0{}_0 + \sum_{i=1,2,3} \Lambda^i{}_0 \Lambda^i{}_0 \\ (\Lambda^0{}_0)^2 &= 1 + \sum_{i=1,2,3} (\Lambda^i{}_0)^2 \quad \geq 1 , \end{aligned} \tag{C.13}$$

so the square of the time-time component of any Λ must always be at least

1, and $\Lambda^0{}_0$ must be either ≥ 1 or ≤ -1. Again, you cannot go continuously from ≥ 1 to ≤ -1, so no elements of $SO(3,1)$ have $\Lambda^0{}_0 < 0$. An element of $O(3,1)$ with $\Lambda^0{}_0 \geq 1$ is called *orthochronous*, and an element with $\Lambda^0{}_0 \leq -1$ is called *non-orthochronous*.

The canonical example of an improper (but orthochronous) element of $O(3,1)$ is the *parity transformation*,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{C.14}$$

which satisfies Eq. (C.6) but has determinant -1. The canonical example of a non-orthochronous (and also improper) transformation is the *time reversal transformation*,

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{C.15}$$

which also satisfies Eq. (C.6) but has $T^0{}_0 = -1$. It turns out that any element of $O(3,1)$ must be an element of $SO(3,1)$, times either the identity (proper orthochronous), P (improper orthochronous), T (improper non-orthochronous), or PT (proper non-orthochronous). The improper or non-orthochronous Lorentz transformations need not be symmetries of nature–in fact, in the standard model, they are not–but it is an axiom of field theory that the elements of $SO(3,1)$ must be symmetries.

C.2 Generators of the Lorentz group

As discussed in Section B.1, each element $\Lambda \in SO(3,1)$ must have associated with it a unitary operator $U(\Lambda)$ which implements it on the Hilbert space, and which represents the group operation,

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)\,. \tag{C.16}$$

For an element infinitesimally close to the identity, it must be possible to expand these operators in a Lie algebra of generators,

$$U(\omega) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + O(\omega^2)\,, \tag{C.17}$$

for some operators $\hat{J}^{\mu\nu}$, antisymmetric in μ, ν. Similarly, there are generators for translations,

$$U(\xi) = \mathbf{1} - i\xi_\mu \hat{P}^\mu \,. \tag{C.18}$$

The $\hat{P}^i$ are also called momentum operators, and the $\hat{J}^{ij}$ are called angular momentum operators.

The commutation relations between the operators $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$ can be worked out by using Eq. (C.16). For instance, consider a translation by a small distance ξ^μ, either preceded or followed by a Lorentz transformation involving $\omega^\nu{}_\alpha$. We will evaluate the difference between the two orders of operation, in two ways.

First, if we translate first and then rotate, then the coordinate is transformed according to

$$\begin{aligned} x'^\mu &= x^\mu + \xi^\mu \,, \\ x''^\mu &= (\delta^\mu{}_\nu + \omega^\mu{}_\nu)(x^\nu + \xi^\nu) \\ &= x^\mu + (\omega^\mu{}_\nu x^\nu) + (\xi^\mu + \omega^\mu{}_\nu \xi^\nu) \,, \end{aligned} \tag{C.19}$$

where the first and second parenthesis represent a rotation and a translation. The result is the same rotation, and a translation by ξ^μ plus an extra piece involving ω and ξ. If the rotation is performed first, we get

$$\begin{aligned} x'^\mu &= x^\mu + \omega^\mu{}_\nu x^\nu \,, \\ x''^\mu &= x^\mu + (\omega^\mu{}_\nu x^\nu) + (\xi^\mu) \,, \end{aligned} \tag{C.20}$$

which is the rotation and the translation just by ξ. The unitary operators for these transformations are,

$$\begin{aligned} U(\omega\xi) &= \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} - i\eta_{\mu\nu}(\xi^\mu + \omega^\mu{}_\alpha \xi^\alpha)\hat{P}^\nu \,, \\ U(\xi\omega) &= \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} - i\eta_{\mu\nu}(\xi^\mu)\hat{P}^\nu \,. \end{aligned} \tag{C.21}$$

The difference of the operators, to second order in the infinitesimals, is

$$U(\omega\xi) - U(\xi\omega) = -i\eta_{\mu\nu}\omega^\mu{}_\alpha \xi^\alpha \hat{P}^\nu \,. \tag{C.22}$$

(There is actually also a second order in ω piece, but it is the same for the two U's and therefore cancels in this difference.)

Alternately, we can say using Eq. (C.16) that

$$\begin{aligned} U(\omega\xi) = U(\omega)U(\xi) &= \left(\mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right)\left(\mathbf{1} - i\eta_{\alpha\beta}\xi^\alpha \hat{P}^\beta\right) \,, \\ U(\xi\omega) = U(\xi)U(\omega) &= \left(\mathbf{1} - i\eta_{\alpha\beta}\xi^\alpha \hat{P}^\beta\right)\left(\mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right) \,, \end{aligned}$$

$$U(\omega\xi) - U(\xi\omega) \quad = \quad -\frac{i^2}{2}\omega_{\mu\nu}\eta_{\alpha\beta}\xi^\alpha \left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\beta\right]. \tag{C.23}$$

Now equating Eq. (C.22) and Eq. (C.23), we learn what the commutator of $\hat{P}$ with $\hat{J}$ must be:

$$-\frac{i^2}{2}\omega_{\mu\nu}\xi_\alpha\left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\alpha\right] = -i\omega_{\mu\nu}\xi_\alpha\eta^{\nu\alpha}\hat{P}^\mu\,. \tag{C.24}$$

This must hold for any antisymmetric $\omega_{\mu\nu}$ and any ξ_α, so (antisymmetrizing over the indices on ω) the operators must satisfy

$$\left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\alpha\right] = i\left(\eta^{\mu\alpha}\hat{P}^\nu - \eta^{\nu\alpha}\hat{P}^\mu\right)\,. \tag{C.25}$$

By a completely analogous but more involved procedure one can also show,

$$\left[\hat{J}^{\mu\nu}\,,\,\hat{J}^{\alpha\beta}\right] = i\left(\eta^{\nu\beta}\hat{J}^{\mu\alpha} + \eta^{\mu\alpha}\hat{J}^{\nu\beta} - \eta^{\mu\beta}\hat{J}^{\nu\alpha} - \eta^{\nu\alpha}\hat{J}^{\mu\beta}\right)\,, \tag{C.26}$$

and (this is simpler)

$$\left[\hat{P}^\mu\,,\,\hat{P}^\nu\right] = 0\,. \tag{C.27}$$

These commutation relations are called the Poincaré algebra.

To make contact with the more familiar generators of rotations and boosts, it is convenient to define

$$\hat{J}_i \quad \equiv \quad \frac{\epsilon_{ijk}}{2}\hat{J}_{jk}\,, \tag{C.28}$$

$$\hat{K}_i \quad \equiv \quad \hat{J}^0{}_i\,, \tag{C.29}$$

which are respectively the generator of rotations about the i axis and of boosts along the i axis, so a rotation by $\vec{\theta}$ is $\exp(-iJ_i\theta_i)$ and a boost by $\vec{v}$ is $\exp(-iK_iv_i)$. They satisfy the commutation relations, following from Eq. (C.26),

$$\left[\hat{J}_i\,,\,\hat{J}_j\right] \quad = \quad i\epsilon_{ijk}\hat{J}_k\,, \tag{C.30}$$

$$\left[\hat{J}_i\,,\,\hat{K}_j\right] \quad = \quad i\epsilon_{ijk}\hat{K}_k\,, \tag{C.31}$$

$$\left[\hat{K}_i\,,\,\hat{K}_j\right] \quad = \quad -i\epsilon_{ijk}\hat{J}_k\,. \tag{C.32}$$

The first expression is the familiar commutator between rotations. The second means that, if a rotation is performed before a boost, the boost will be in a different direction than before the rotation is performed, which is intuitively clear. The third result is more surprising; the commutator of two boosts is a rotation. More importantly, the sign is opposite on the last result than on the previous two.

C.3 Representations of the Lorentz group

Just as for an internal symmetry, an $SO(3,1)$ transformation will carry a field to a linear combination of fields, so the fields must transform under representations of the group. The difference is that the transformed field will be at the Lorentz transformed point:

$$U(\omega)\varphi_a(x)U^*(\omega) = D^{-1}_{ab}(\omega)\varphi_b(\Lambda x)\,, \tag{C.33}$$

with $D^{-1}(\omega) = D(-\omega)$ an ω dependent matrix in the space of fields. The fields can be chosen to block-diagonalize the matrix D into irreducible representations of the Lorentz group. For instance, in QED, the components of the gauge potential A^μ mix with each other under Lorentz transformations, but they never mix with the different spin components of the electron e_i; so there is one "block" of D which mixes the A^μ and an independent block mixing the e_i. Our goal now is to find the possible structures D can take.

Just as for internal symmetries, there are two very simple irreducible representations, which are also physically important. One is the trivial (singlet) representation,

$$U(\omega)\phi(x)U^*(\omega) = \phi(\Lambda x)\,, \tag{C.34}$$

which for $SO(3,1)$ is called the *scalar representation*. Lorentz symmetry demands that the Lagrangian be a Lorentz scalar. The other is the *vector representation*, for which the field index is a four-vector index and the representation matrix is Λ itself:

$$U(\omega)A^\mu(x)U^*(\omega) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)\,. \tag{C.35}$$

A representation is determined by a set of matrices $\mathcal{J}^{\mu\nu}$ with the same commutation relations as the $\hat{J}^{\mu\nu}$. That is, the matrix D_{ab} must be of the form

$$D_{ab}(\omega) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}_{ab}\right)\,, \tag{C.36}$$

with the exponentiation interpreted as matrix exponentiation with a, b the matrix indices, and $\mathcal{J}^{\mu\nu}$ satisfying

$$\left[\mathcal{J}^{\mu\nu}\,,\,\mathcal{J}^{\alpha\beta}\right] = i\left(\eta^{\nu\beta}\mathcal{J}^{\mu\alpha} + \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} - \eta^{\mu\beta}\mathcal{J}^{\nu\alpha} - \eta^{\nu\alpha}\mathcal{J}^{\mu\beta}\right)\,. \tag{C.37}$$

The problem of finding representations is the problem of finding all sets of matrices with this algebra.

It is believed that only field theories containing finite numbers of fields are well defined. Therefore we need only look for finite-dimensional representations of $SO(3,1)$. The classification of the representations is made easier by the following convenient property of the group. Using Eq. (C.30)–Eq. (C.32), one can show that the operators

$$\hat{\mathcal{L}}_i \equiv \frac{\hat{J}_i + i\hat{K}_i}{2}, \qquad \hat{\mathcal{R}}_i \equiv \frac{\hat{J}_i - i\hat{K}_i}{2}, \tag{C.38}$$

satisfy the commutation relations,

$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{L}}_j\right] = i\epsilon_{ijk}\hat{\mathcal{L}}_k\,, \tag{C.39}$$

$$\left[\hat{\mathcal{R}}_i\,,\,\hat{\mathcal{R}}_j\right] = i\epsilon_{ijk}\hat{\mathcal{R}}_k\,, \tag{C.40}$$

$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{R}}_j\right] = 0\,. \tag{C.41}$$

Therefore the generators of $SO(3,1)$ can be split into two subsets which commute with each other, and each satisfy the same commutation relations as the group $SU(2)$. This group is familiar as the group of rotations and its representations are well known; they are the spin-zero representation, the spin-half representation, the spin-one representation, and so forth. A general irreducible representation can be described by its transformation properties under $\hat{\mathcal{L}}$ and under $\hat{\mathcal{R}}$, *eg*, spin $m/2$ under $\hat{\mathcal{L}}$ and spin $n/2$ under $\hat{\mathcal{R}}$.

Only four representations will be of any interest to us in studying the standard model, because it turns out that only four representations can participate in renormalizable interactions in a theory satisfying the basic principles laid out in Section 1.2.

The first of these is the scalar representation already introduced. The scalar representation transforms as $(0,0)$, that is, as spin-zero under $\hat{\mathcal{L}}$ and spin-zero under $\hat{\mathcal{R}}$. The Lie algebra representations are $\hat{\mathcal{J}}^{\mu\nu} = 0$ and the transformation matrix $D = 1$ is the identity.

The second common representation is the vector representation, which transforms as $(\frac{1}{2}, \frac{1}{2})$. The Lie algebra is represented as $\mathcal{J}^{\mu\nu}{}_{\alpha\ \beta} = -i(\eta^{\mu\alpha}\eta^{\nu}_{\beta} - \eta^{\mu}_{\beta}\eta^{\nu\alpha})$, and $D = \Lambda$, as already discussed. Note that for both of these representations, the matrix D is always real; therefore it is consistent to consider real valued scalar or vector fields.

The other two interesting representations are called *spinor representations*, and consist of two fields which mix with each other under Lorentz transformations. Since these are probably less familiar to the reader and are in some ways more complicated than the scalar and vector representations, we will discuss them at length in the next section.

C.4 Spinors and the Dirac algebra

We now introduce the other two physically important representations of the Lorentz group, the left and right handed spinor representations. A field transforming in one of these can be rewritten in terms of the other, and it is convenient to combine them together using *Majorana notation*, which we will also introduce and which we use throughout this book.

C.4.1 Spinor representations

The simplest nontrivial matrices which satisfy Eq. (C.39) are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.42}$$

which satisfy the commutation relation

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{C.43}$$

Therefore, if the matrices representing $\hat{\mathcal{L}}_i$ and $\hat{\mathcal{R}}_i$ are $\sigma_i/2$ and 0 respectively, we get a representation of the Lorentz algebra. Inverting Eq. (C.38), rotations and boosts are implemented by the matrices,

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = -i\frac{\sigma_i}{2}, \qquad \text{(Left handed spinor)} \tag{C.44}$$

which it is easy to show satisfy Eq. (C.30) through Eq. (C.32).

Therefore, a pair of fields $\psi_a, a = 1, 2$ can transform under Lorentz transformations according to

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-i(r_i - ib_i)\sigma_i/2)], \tag{C.45}$$

where r_i, b_i are the amount of rotation and boost performed, as introduced in Eq. (C.9). The two fields ψ_a are generally referred to as the components of a single *spinor field* with a the *spinor index*, which is almost always suppressed by writing ψ and D in matrix notation (ψ as a column vector, D as a matrix). Such a spinor field is called a *left handed Weyl spinor* ψ_L.

Alternately, $\mathcal{R}_i$ could be represented by the Pauli matrices and $\mathcal{L}_i$ by 0's,

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = i\frac{\sigma_i}{2}, \qquad \text{(Right handed spinor)} \tag{C.46}$$

in which case a Lorentz transform acts on ψ via

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-i(r_i + ib_i)\sigma_i/2)]. \tag{C.47}$$

A field transforming this way is called a *right handed Weyl spinor* ψ_R.

Since the matrices D we just constructed are in general complex, a spinor ψ_L or ψ_R must be a pair of complex fields. We can ask how the complex conjugate of ψ_L transforms. Because complex conjugation flips the i in front of $\mathcal{K}$ in Eq. (C.44), the answer is that it transforms as a right handed Weyl spinor. More properly, defining the matrix

$$\epsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{satisfying} \qquad \epsilon\sigma_i^* = -\sigma_i\epsilon\,, \tag{C.48}$$

we see that ϵ times the conjugate of ψ_L transforms according to,

$$\begin{aligned} U(-\omega)\epsilon\psi_L^* U^*(-\omega) &= \epsilon\left(\exp\left[-i(r_i - ib_i)\frac{\sigma_i}{2}\right]\psi_L\right)^* \\ &= \epsilon\exp\left[+i(r_i + ib_i)\frac{\sigma_i^*}{2}\right]\psi_L^* \\ &= \exp\left[-i(r_i + ib_i)\frac{\sigma_i}{2}\right]\epsilon\psi_L^*\,, \end{aligned} \tag{C.49}$$

which is precisely the transformation rule for a right handed Weyl spinor. Similarly, $-\epsilon\psi_R^*$ transforms as a left handed Weyl spinor, and $-\epsilon(\epsilon\psi_L^*)^* = \psi_L$ transforms as a left handed Weyl spinor again. Both the field and its complex conjugate will typically appear in the Lagrangian so it is important to have a notation which can deal with each. Whether we consider the left or right handed version as the field rather than the conjugated object is a matter of convention.

C.4.2 Weyl, Majorana, Dirac

There are two common notational ways of dealing with the fact that a field can be written either as a left or a right handed spinor.

One, called *Weyl notation*, expresses the fields as two component objects, and then specifies whether one is referring to ψ_L or to its right handed conjugate $\epsilon\psi_L^*$ by using either an undotted or a dotted index: $\psi_\alpha = \psi_L$ and $\psi^{\dot\alpha} = \epsilon\psi_L^*$. (Indices are raised and lowered using ϵ and dotted and undotted according to whether they are conjugated.) This notation is common in the supersymmetry and string theory literature.

An alternative which we will use, *Majorana notation*, writes a single *four* component field ψ_M, defined as

$$\psi_M = \begin{pmatrix} \psi_L \\ \epsilon\psi_L^* \end{pmatrix}, \tag{C.50}$$

that is, ψ_M redundantly records both the left handed and the right handed ways of writing the field. The individual pieces can be accessed separately by using the projection operators

$$P_L \equiv \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_R \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{C.51}$$

The action of rotations and boosts on ψ_M are respectively,

$$\mathcal{J}_i = \begin{pmatrix} \frac{\sigma_i}{2} & 0 \\ 0 & \frac{\sigma_i}{2} \end{pmatrix}, \qquad \mathcal{K}_i = \begin{pmatrix} \frac{-i\sigma_i}{2} & 0 \\ 0 & \frac{i\sigma_i}{2} \end{pmatrix}. \tag{C.52}$$

If a left-handed spinor transforms nontrivially under an internal symmetry group, then since the right-handed version involves complex conjugation, the right-handed version $\epsilon\psi_L^*$ transforms under the conjugate representation. In particular, if ψ_L has charge q under a $U(1)$ symmetry and is in the fundamental representation of an $SU(N)$ symmetry, then $\epsilon\psi_L^*$ has charge $-q$ and transforms under the anti-fundamental representation of $SU(N)$. One must keep this in mind when constructing Lagrangians out of Majorana spinors.

In QED and QCD, if we write the spinor fields as left-handed objects, the fields form pairs with conjugate symmetry transformation properties. For instance, in QED, there is a field $\mathcal{E}_L$ which is charge -1 under $U_{em}(1)$, called the left-handed electron, and a field $-\epsilon E_R^*$ which is charge 1 under $U_{em}(1)$, called the left-handed positron. In this case it is most convenient to think of the latter as the conjugate of a right-handed field with charge -1, E_R, called the right-handed electron, and to combine them together in a single 4-component object called a *Dirac spinor*, $e = [\mathcal{E}_L \; E_R]^T$.

The Lorentz transformation properties of Majorana and Dirac spinors are the same. The two distinctions are that the upper and lower components of a Dirac spinor generally have the same transformation properties under internal symmetries, while for Majorana spinors they have conjugate transformation properties; and the upper and lower components of a Dirac spinor are independent, while for a Majorana spinor they are redundant notations for the same field.

C.4.3 Tensor products of spinors

Since the Lagrangian must be a Lorentz scalar, it must be a sum of terms even in spinorial fields. Therefore we need to know how products of two spinor fields transform. We will only consider the combination of a spinor field ψ_1 with the complex conjugate of another, $\psi_2^\dagger$. This is sufficient for Majorana spinors because $\psi_2^T\psi_1$ can be re-expressed in terms of $\psi_2^\dagger\psi_1$, and it suffices for Dirac spinors with internal symmetries because only such combinations are invariant under the internal symmetries.

The Hermitian conjugate of a spinor field ψ transforms as

$$U(-\omega)\psi^\dagger U^*(-\omega) = (D(\omega)\psi)^\dagger = \psi^\dagger D^\dagger(\omega)\,. \qquad \text{(C.53)}$$

The $\mathcal{J}_i$ are Hermitian, but the $\mathcal{K}_i$ are anti-Hermitian, so $D(\omega)$ is not in general unitary. Therefore $\psi^\dagger$ does not have the inverse transformation property of ψ. However, there is a Hermitian, unit determinant matrix β,

$$\beta \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\,, \qquad \beta\mathcal{J}_i = \mathcal{J}_i\beta\,, \quad \beta\mathcal{K}_i = -\mathcal{K}_i\beta\,, \qquad \text{(C.54)}$$

which flips the sign of $\mathcal{K}$ but not $\mathcal{J}$ when commuted across $D^\dagger$, so $D^\dagger\beta = \beta D^{-1}$. Therefore, defining $\overline{\psi} = \psi^\dagger\beta$, called the *Dirac conjugate* of ψ,

$$U(-\omega)\overline{\psi}\beta U^*(-\omega) = \psi^\dagger D^\dagger(\omega)\beta = \psi^\dagger\beta D^{-1}(\omega) = \overline{\psi}D^{-1}(\omega)\,, \qquad \text{(C.55)}$$

so $\overline{\psi}$ has the inverse transformation property of ψ.

Since ψ has 4 components, there are sixteen independent 4×4 matrices Γ which can be used to combine spinors, $\overline{\psi}_2\Gamma\psi_1$. These can all be gotten from four such matrices, called the *gamma matrices* γ^μ, given in Eq. (1.87). These satisfy anticommutation relations called the *Clifford algebra*,

$$\left\{\gamma^\mu\,,\,\gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbf{1}\,. \qquad \text{(C.56)}$$

The matrices $\mathcal{J}^{\mu\nu}$ can be expressed in terms of the gamma matrices:

$$\mathcal{J}^{\mu\nu} = \frac{-i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right]\,, \qquad \text{(C.57)}$$

which together with Eq. (C.56) is enough to prove that $\mathcal{J}^{\mu\nu}$ satisfies the Lorentz algebra, Eq. (C.37). Further, these relations ensure that

$$\left[\mathcal{J}^{\mu\nu}\,,\,\gamma^\alpha\right] = i\left(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu\right)\,, \qquad \text{(C.58)}$$

from which it follows that

$$D^{-1}(\omega)\gamma^\mu D(\omega) = \Lambda^\mu{}_\nu\gamma^\nu\,. \qquad \text{(C.59)}$$

Therefore, while the combination $\overline{\psi}_2\psi_1$ is a scalar,

$$U(-\omega)\overline{\psi}_2\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}(\omega)D(\omega)\psi = \overline{\psi}_2\psi_1 \quad \text{is a scalar}\,, \qquad \text{(C.60)}$$

the combination $\overline{\psi}_2\gamma^\mu\psi_1$ is a vector,

$$U(-\omega)\overline{\psi}_2\gamma^\mu\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}(\omega)\gamma^\mu D(\omega)\psi_1 = \Lambda^\mu{}_\nu\psi_2\gamma^\nu\psi_1 \quad \text{is a vector}\,. \qquad \text{(C.61)}$$

Similarly, defining $\sigma^{\mu\nu} = 2i\mathcal{J}^{\mu\nu}$, the combination

$$U(-\omega)\overline{\psi}_2\sigma^{\mu\nu}\psi_1 U^{-1}(-\omega) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\overline{\psi}_2\sigma^{\alpha\beta}\psi_1 \tag{C.62}$$

is a rank-2 antisymmetric tensor.

Next, define

$$\gamma^5 = \gamma_5 \equiv \frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = -i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{C.63}$$

where the latter follows from the anti-commutation of the distinct gamma matrices. We have that

$$\begin{aligned} U(-\omega)\overline{\psi}_2\gamma^5\psi_1 U^{-1}(-\omega) &= \frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\overline{\psi}_2 D^{-1}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta D\psi_1 \\ &= \frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho\Lambda^\alpha{}_\kappa\Lambda^\beta{}_\zeta\overline{\psi}_2\gamma^\sigma\gamma^\rho\gamma^\kappa\gamma^\zeta\psi_1 \\ &= (\mathrm{Det}\,\Lambda)\ \frac{i}{24}\epsilon_{\sigma\rho\kappa\zeta}\overline{\psi}_2\gamma^\sigma\gamma^\rho\gamma^\kappa\gamma^\zeta\psi_1 \\ &= (\mathrm{Det}\,\Lambda)\ \overline{\psi}_2\gamma^5\psi_1\,. \end{aligned} \tag{C.64}$$

Therefore $\overline{\psi}_2\gamma^5\psi_1$ is a *pseudoscalar*, a scalar under $SO(3,1)$ which flips sign under parity transformations. Finally, the quantity $\overline{\psi}_2\gamma^\mu\gamma^5\psi$ transforms as a pseudovector,

$$U(-\omega)\overline{\psi}_2\gamma^\mu\gamma^5\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}\gamma^\mu\gamma^5 D\psi_1 = (\mathrm{Det}\,\Lambda)\Lambda^\mu{}_\nu\overline{\psi}_2\gamma^\nu\gamma^5\psi_1\,. \tag{C.65}$$

Since this gives $1+4+6+4+1 = 16$ independent contractions, the above are exhaustive; any other matrix sandwiched between $\overline{\psi}_2$ and ψ_1 must be a linear combination of $\mathbf{1}$, γ^μ, $\sigma^{\mu\nu}$, $\gamma^\mu\gamma^5$, and γ^5.

The choice of matrices made above is called the chiral basis and is convenient because the right and left handed components of ψ factorize. However, multiplying ψ by an arbitrary unitary matrix S and all matrices by $\Gamma \to S\Gamma S^{-1}$ leaves the theory unchanged. While the explicit expressions for the matrices are obviously changed, certain relations are not, and are therefore particularly valuable. In particular, the Clifford algebra, Eq. (C.56), the relations Eq. (C.57), Eq. (C.58), Eq. (C.59), the definition Eq. (C.63) of γ^5 in terms of the other γ matrices, and the relations between the projection operators and γ^5,

$$P_L = \frac{1+\gamma^5}{2}\,, \qquad P_R = \frac{1-\gamma^5}{2}\,, \tag{C.66}$$

are basis independent and should therefore be sufficient to evaluate any invariant quantities.