Physics 731 Homework 1

Due 15 January 2007

1 Pauli-Lubanski vector

Prove from the definition of the Pauli-Lubanski vector

$$\hat{W}^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{P}_\nu \hat{J}_{\alpha\beta} \tag{1}$$

and the commutation relations for $\hat{P}^\mu$ and $\hat{J}^{\mu\nu}$ that $\hat{W}^\mu$ transforms as a (pseudo)vector,

$$\begin{aligned} \left[\hat{W}^\mu\,,\,\hat{P}^\nu\right] &= 0\,, & (2)\\ \left[\hat{W}^\mu\,,\,\hat{J}^{\nu\alpha}\right] &= i\left(\eta^{\mu\alpha}\hat{W}^\nu - \eta^{\mu\nu}\hat{W}^\alpha\right)\,. & (3) \end{aligned}$$

Use these to show that $\hat{W}^2$ is a Casimir operator,

$$\left[\hat{W}^2\,,\,\hat{P}^\mu\right] = 0 = \left[\hat{W}^2\,,\,\hat{J}^{\mu\nu}\right]. \tag{4}$$

2 J and K

Verify from the $J^{\mu\nu}$ commutation relations that the objects

$$J_i \equiv \frac{\epsilon_{ijk}}{2}J_{jk} \qquad \text{and} \qquad K_i \equiv J^0{}_i \tag{5}$$

satisfy the commutation relations

$$\begin{aligned} \left[J_i\,,\,J_j\right] &= i\epsilon_{ijk}J_k\,, & (6)\\ \left[J_i\,,\,K_j\right] &= i\epsilon_{ijk}K_k\,, & (7)\\ \left[K_i\,,\,K_j\right] &= -i\epsilon_{ijk}J_k\,. & (8) \end{aligned}$$

Argue by counting the number of independent objects that the behavior of J_i and K_i is enough to fully specify the behavior of $J_{\mu\nu}$. (Following common conventions, Roman indices run over space 1,2,3 with positive metric. One advantage of the $[-+++]$ metric is that one needn't distinguish between upper and lower space indices.)

Now show (this is easy) that

$$L_i \equiv \frac{J_i + iK_i}{2} \qquad \text{and} \qquad R_i \equiv \frac{J_i - iK_i}{2} \tag{9}$$

satisfy the commutation relations

$$\begin{aligned}
\Big[L_i\,,\,L_j\Big] &= i\epsilon_{ijk}L_k\,, &(10)\\
\Big[R_i\,,\,R_j\Big] &= i\epsilon_{ijk}R_k\,, &(11)\\
\Big[L_i\,,\,R_j\Big] &= 0\,. &(12)
\end{aligned}$$

Finally, verify that the choices

$$J_i = \frac{\sigma_i}{2}\,, \qquad K_i = \frac{\pm i\sigma_i}{2} \tag{13}$$

satisfy the relations, Eq. (6) through Eq. (8), for either sign $\pm = +, -$.

3 Four-vector

Consider two spinor fields χ_α and ψ_α. Verify that under Lorentz transformation the combination

$$A^\mu \equiv \chi^\dagger_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\alpha}\,\psi_\alpha \tag{14}$$

transforms as a 4-vector under Lorentz transformations,

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu\,. \tag{15}$$

Hint: you need only consider infinitesimal Lorentz transformations and work to first order in rotations r_i and boosts b_i. If you have trouble, do brute force analysis of each component rotation and boost.