Physics 731 Homework 3

Due 31 January 2007

1 Susy representation on massless fields

Consider a massless field $|p, \alpha\rangle$, where $p^\mu = p(1, 0, 0, 1)$ (so the p_z component equals the energy). The susy algebra implies that

$$\left(Q_\alpha \bar{Q}_{\dot{\beta}} + \bar{Q}_{\dot{\beta}} Q_\alpha\right) |p, \alpha\rangle = 2\sigma^\mu_{\alpha\dot{\beta}} p_\mu |p, \alpha\rangle \, . \tag{1}$$

Find the explicit form of the matrix $p_\mu \sigma^\mu_{\alpha\dot{\beta}}$ for the case at hand, and argue from its structure that it is perfectly consistent for $\bar{Q}$ and one of the components of Q to annihilate the state $|p, \alpha\rangle$, but that, in this case, the remaining component of Q cannot annihilate the state.

Suppose that this is so; $\bar{Q}$ and one component of Q annihilate the state. Which component of Q does not annihilate the state? If $W^\mu |p, \alpha\rangle = \alpha p^\mu |p, \alpha\rangle$ (so the state has chirality α), what is the chirality of the state $|\psi\rangle \equiv Q|p, \alpha\rangle$? Show that Q and one component of $\bar{Q}$ annihilate $|\psi\rangle$, but one component of $\bar{Q}$ acts on it to return a multiple of $|p, \alpha\rangle$. Therefore, you have found a multiplet with two states separated by a half-integer in chirality.
[Hint: in dealing with W^μ, you already know on general grounds that $W^\mu |p\rangle = \lambda p^\mu |p\rangle$. Therefore it is sufficient to consider the action of $W^3 = pJ_{12}$. Therefore you just need to think about how J_{12} (z-rotations) commute with the relevant Q operator.]

2 Susy algebra with Grassman parameters

Using the fact that the supersymmetry parameter ξ_α (and similarly $\bar{\xi}_{\dot{\alpha}}$) commutes with all bosonic and anticommutes with all fermionic quantities, *i.e.*

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, \bar{\xi}^{\dot{\beta}}\} = \{\xi^\alpha, Q^\beta\} = ... = [P^\mu, \xi^\alpha] = 0 \, , \tag{2}$$

show that the supersymmetry algebra can be written solely in terms of commutators when introducing the scalars $\xi Q = \xi^\alpha Q_\alpha$ and $\bar{\xi}\bar{Q} = \bar{\xi}_{\dot{\alpha}} Q^{\dot{\alpha}}$. Verify that the susy algebra now takes the form

$$\begin{aligned} [\xi Q, \bar{\eta}\bar{Q}] &= 2\xi\sigma^\mu\bar{\eta}\, P_\mu \\ [\xi Q, \eta Q] &= [\bar{\xi}\bar{Q}, \bar{\eta}\bar{Q}] = 0 \\ [P^\mu, \xi Q] &= [P^\mu, \bar{\xi}\bar{Q}] = 0 \, . \end{aligned} \tag{3}$$

3 Double SUSY variation

We defined the action of supersymmetry on a field Φ (not necessarily a scalar) as the commutator

$$\delta_\xi \Phi = -i[\Phi, \xi Q + \bar{\xi}\bar{Q}]\,. \tag{4}$$

With the help of Eq. (3) show that

$$(\delta_\xi \delta_\eta - \delta_\eta \delta_\xi)\Phi = 2i(\eta\sigma^\mu \bar{\xi} - \xi\sigma^\mu \bar{\eta})\partial_\mu \Phi \tag{5}$$

and convince yourself that what we derived in class (modulo minus signs) was indeed the "closure of the algebra" on a field multiplet (A, ψ, F), which is called the *chiral multiplet* by convention. It should also be clear now that this is the smallest possible representation of an $\mathcal{N} = 1$ susy field multiplet in four dimensions.
Hint: Recall that $[\Phi, P_\mu] = i\partial_\mu \Phi$ generates translations.

4 Spinor relations

Just to make sure we follow the spinor manipulations used in class, verify the following spinor and sigma-matrix identities:

$$\begin{aligned}
\xi\eta \equiv \xi^\alpha \eta_\alpha &= \eta\xi \equiv \eta^\alpha \xi_\alpha\,, && (6)\\
\bar{\xi}\bar{\eta} \equiv \bar{\xi}_{\dot\alpha}\bar{\eta}^{\dot\alpha} &= \bar{\eta}\bar{\xi} \equiv \bar{\eta}_{\dot\alpha}\bar{\xi}^{\dot\alpha}\,, && (7)\\
(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta_\alpha{}^\beta\,, && (8)\\
\sigma^\mu_{\alpha\dot\alpha}\bar{\sigma}^{\dot\beta\beta}_\mu &= -2\delta_\alpha{}^\beta \delta_{\dot\alpha}{}^{\dot\beta}\,. && (9)
\end{aligned}$$

(Note that these are to be verified in the $[-{+}{+}{+}]$ metric. The sign on the last will differ in the mostly negative metric [why?].)