Physics 731 Homework 6

Due 2 April 2007 (?)

Notation

As a note, because we use a mostly positive metric, our Dirac matrices differ from those in Sohnius by a factor of i. That is

$$\Gamma^{\mu}\,(\text{here}) = i\Gamma^{\mu}\,(\text{Sohnius}) \ .$$

Also, we use

$$\gamma_5\,(\text{here}) = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = -i\gamma_5\,(\text{Sohnius}) \ .$$

That explains some differences in factors from equations in Sohnius.

1 The Majorana Matrix

We defined a "Majorana matrix" X (as in Sohnius §14.1 and §A.7) by

$$X\Gamma^{\mu}X^{-1} = \pm\left(\Gamma^{\mu}\right)^{*} \tag{1}$$

(with the sign depending on the dimensionality) such that a Majorana spinor satisfies $\psi = X\psi^{*}$.

Consider a unitary change of basis for the Dirac matrices, $\Gamma^{\mu} \to U\Gamma^{\mu}U^{\dagger}$. For (1 to hold in the second basis as well as the first, find the transformation rule for X under the change of basis.

For the Majorana condition to be compatible with Lorentz covariance, X must satisfy

$$X\Sigma^{\mu\nu}X^{-1} = -\left(\Sigma^{\mu\nu}\right)^{*} \tag{2}$$

for the Lorentz generators $\Sigma^{\mu\nu} = -i\Gamma^{\mu\nu}$. Prove (2) from (1) and also show that (2) is unchanged by a unitary change of basis.

Finally, consider a basis in which all the Dirac matrices are real (so Γ^0 is antisymmetric and all the spatial matrices are symmetric). Find X for the $+$ (as in 6 and 10 dimensions) and $-$ (as in 4 dimensions) signs in (1) in this basis.

This problem is largely based on problem B.1 in *String Theory* by Polchinski.

2 Dimensional Reduction of SUSY Variations

Consider the 6 dimensional supersymmetric gauge theory from §14.2 of Sohnius. As you've learned, trivial dimensional reduction of this theory on a torus along x^4, x^5 (meaning that none of the fields depend on the torus directions) is just the $\mathcal{N} = 2$ supersymmetric gauge theory in 4 dimensions. For notation, we will use M, N to represent all 6 dimensions and μ, ν to represent the 4 noncompact dimensions. Note that we're not making Sohnius's unusual choice of x^5, x^6 for the extra two dimensions.

First, with the trivial dimensional reduction, show that the 6D gaugino supersymmetry transformation (from Sohnius (14.22))

$$\delta\lambda = \frac{1}{2} i F_{MN} \Sigma^{MN} \zeta \tag{3}$$

reduces to the gaugino variation of the 4D $\mathcal{N} = 2$ theory (from Sohnius (12.6))

$$\delta\lambda_i = \frac{1}{2} i F_{\mu\nu} \Sigma^{\mu\nu} \zeta_i + \gamma^\mu \nabla_\mu \left(M + i\gamma_5 N \right) \epsilon_{ij} \zeta_j - \gamma_5 [M, N] \zeta_i \ . \tag{4}$$

You'll find the spinor and Dirac matrix decomposition in Sohnius §14.2 useful.

Next, focus on a $U(1)$ gauge theory in 6D, so all the commutators can vanish. In this case, it is possible to dimensionally reduce with $F_{45} = B \neq 0$ using an ansatz

$$A_4 = -\frac{1}{2} B x^5 + N(x^\mu) \ , \quad A_5 = \frac{1}{2} B x^4 + M(x^\mu) \ . \tag{5}$$

Reduce the supersymmetry transformations from 6 to 4 dimensions again and show that B appears as a D-term, as in Sohnius (12.13b) (at least up to a factor of γ_5).

3 More Mass-Splitting Rules

Reconsider Martin's proof of the tree-level mass-splitting relation

$$\mathrm{Tr}\left(\mathbf{m}_S^2\right) - 2\mathrm{Tr}\left(\mathbf{m}_F^\dagger \mathbf{m}_F\right) + 3\mathrm{Tr}\left(\mathbf{m}_V^2\right) = 0 \tag{6}$$

and show that it holds separately for each representation (charge sector) of the unbroken gauge group.

4 Another Index

Read §6 of the Witten article from the reading (*Nuclear Physics* **B**202, pg 253) on Abelian gauge theories. Let the Fayet-Iliopoulos term vanish, and calculate $\mathrm{Tr}[(-1)^F CP]$, following the same logic as the reading.

Further Reading

If you're curious, a more recent calculation of an index appears in `hep-th/0208032`, which also includes more about magnetic fields in compactifications.