Physics 742 Homework 1

1 Lorentz transform warmup

Consider a boost by amount b_1 in the x-direction. Take b_1 to be small and write out $\Lambda^\mu{}_\nu = \exp \omega^\mu{}_\nu$ to *second* order in b_1.

Now consider a boost by b_2 in the y-direction. Again, write out the matrix form of $\Lambda^\mu{}_\nu$ to second order in b_2.

Find the product $\Lambda(b_1)\Lambda(b_2)$ and the product $\Lambda(b_2)\Lambda(b_1)$, to second order in b's. Show that the difference is of order $b_1 b_2$, and looks like an $\omega^\mu{}_\nu$ which generates a rotation. What axis is the rotation about?

2 Commutation relations

Show that the commutation relations

$$\Big[J_i\,,\,J_j\Big] = i\epsilon_{ijk}J_k\,, \tag{1}$$

$$\Big[J_i\,,\,K_j\Big] = i\epsilon_{ijk}K_k\,, \tag{2}$$

$$\Big[K_i\,,\,K_j\Big] = -i\epsilon_{ijk}J_k\,, \tag{3}$$

together with the definitions

$$L_i \equiv \frac{J_i + iK_i}{2}\,, \qquad R_i \equiv \frac{J_i - iK_i}{2} \tag{4}$$

give rise to the commutation relations

$$\Big[L_i\,,\,L_j\Big] = i\epsilon_{ijk}L_k\,, \tag{5}$$

$$\Big[R_i\,,\,R_j\Big] = i\epsilon_{ijk}R_k\,, \tag{6}$$

$$\Big[L_i\,,\,R_j\Big] = 0\,. \tag{7}$$

3 Majorana identities

Using the following relations for γ matrices,

$$\beta\gamma_\mu^\dagger = -\gamma_\mu\beta\,, \tag{8}$$

$$C\gamma_\mu^T = -\gamma_\mu C\,, \tag{9}$$

as well as

$$\bar{\psi}_1 = \psi_1^\dagger \beta\,, \tag{10}$$

$$\bar{\psi}_1^T = -C\psi_1\,, \tag{11}$$

$$\psi_1^T C = \bar{\psi}_1\,, \tag{12}$$

prove these useful relations for Majorana spinors ψ_1, ψ_2,

$$\begin{aligned}
\bar{\psi}_1\psi_2 &= +\,\bar{\psi}_2\psi_1\,,\\
\bar{\psi}_1\gamma^5\psi_2 &= +\,\bar{\psi}_2\gamma^5\psi_1\,,\\
\bar{\psi}_1\gamma^\mu\psi_2 &= -\,\bar{\psi}_2\gamma^\mu\psi_1\,,\\
\bar{\psi}_1\gamma^\mu\gamma^5\psi_2 &= +\,\bar{\psi}_2\gamma^\mu\gamma^5\psi_1\,,\\
\bar{\psi}_1[\gamma^\mu,\gamma^\nu]\psi_2 &= -\,\bar{\psi}_2[\gamma^\mu,\gamma^\nu]\psi_1\,.
\end{aligned}$$

Hint: you can reverse the order of the operators by transposing: $\bar{\psi}_1\gamma_\mu\psi_2 = -\psi_2^T\gamma_\mu^T\bar{\psi}_1^T$. The $-$ sign is from the anticommutation of fermionic operators.

Next, show that

$$\begin{aligned}
\left(\bar{\psi}_1\psi_2\right)^\dagger &= +\,\bar{\psi}_1\psi_2\,,\\
\left(\bar{\psi}_1\gamma^5\psi_2\right)^\dagger &= -\,\bar{\psi}_1\gamma^5\psi_2\,,\\
\left(\bar{\psi}_1\gamma^\mu\psi_2\right)^\dagger &= +\,\bar{\psi}_1\gamma^\mu\psi_2\,,\\
\left(\bar{\psi}_1\gamma^\mu\gamma^5\psi_2\right)^\dagger &= -\,\bar{\psi}_1\gamma^\mu\gamma^5\psi_2\,,\\
\left(\bar{\psi}_1[\gamma^\mu,\gamma^\nu]\psi_2\right)^\dagger &= +\,\bar{\psi}_1[\gamma^\mu,\gamma^\nu]\psi_2\,.
\end{aligned}$$

Hint: Hermitian conjugation reverses the order of operators and daggers them. The matrix β is Hermitian, $\beta^\dagger = \beta$. You will also need the relations you found in the first half.

Use these to justify the requirements on the coefficients A, B, C, D, and E mentioned in the book under Eq. (1.102).

4 Scalars and symmetries

The kinetic term $\frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i$ for N real scalar fields is invariant under a symmetry $\varphi_i \to \mathcal{O}_{ij}\varphi_j$, where $\mathcal{O}^\top\mathcal{O} = 1,\ i,j = 1,...,N$. These form the group of $N\times N$ real orthogonal matrices $\mathcal{O}(N)$.

1. Write down the most general renormalizable Lagrangian for two real scalar fields, φ_1 and φ_2, subject to the discrete symmetries $(\varphi_1, \varphi_2) \to (-\varphi_1, \varphi_2)$ and $(\varphi_1, \varphi_2) \to (\varphi_1, -\varphi_2)$.

2. Re-express this Lagrangian in terms of the complex variables $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ and $\psi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$.

3. The group $\mathcal{O}(2)$ is equivalent to the group $U(1)$. If the $\mathcal{O}(2)$ transformations are written
$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \mathcal{O}(\theta) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \qquad \mathcal{O}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$
find the transformation rules for ψ and ψ^*.

4. What further restrictions are placed on the Lagrangian by requiring that it be $\mathcal{O}(2)$ invariant (including interaction terms)?

   Write the resulting Lagrangian in terms of both the variables (φ_1, φ_2) and (ψ, ψ^*).

5 Adjoint representation

Any matrices T_a satisfying the Lie algebra of a group,

$$\Big[T_a\,,\,T_b\Big] = i f_{abc} T_c\,, \tag{13}$$

generate a representation of the group. This problem shows that the structure functions *themselves* provide one such set of matrices. Define $F^a{}_{bc} = -i f_{abc}$ and consider the first index a to be a label and the second and third indices to be a row and column position.

Using SU(2), because it is simpler, write out the three 3×3 matrices ϵ^1, ϵ^2, and ϵ^3. (For SU(2), the structure function $f_{abc} = \epsilon_{abc}$.) Verify that these matrices in fact satisfy the commutation relations of the Lie algebra, that is, that

$$\Big[\epsilon^a\,,\,\epsilon^b\Big] = i\epsilon_{abc}\epsilon^c\,. \tag{14}$$

Next, prove the Jacobi identity,

$$\Big[\Big[T^a\,,\,T^b\Big]\,,\,T^c\Big] + \Big[\Big[T^b\,,\,T^c\Big]\,,\,T^a\Big] + \Big[\Big[T^c\,,\,T^a\Big]\,,\,T^b\Big] = 0\,, \tag{15}$$

which is just a consequence of writing out every term longhand and canceling like terms. What condition does the Jacobi identity imply on the coefficients f_{abc}?

Finally, show that the antisymmetry of the f_{abc}, together with the Jacobi identity, proves that

$$\left[F^a \, , \, F^b\right] = i f_{abc} F^c \tag{16}$$

holds in any group. Therefore the structure functions themselves provide a representation, called the *adjoint* representation.