Physics 742 Homework 2

1 Disallowed Lagrangian Terms

Explain as briefly as possible why each of the following terms is not a permissible term in the Lagrangian density $\mathcal{L}(x)$ of the Standard Model. Fields are evaluated at point x unless indicated otherwise.

1. $\bar{L}P_L L + \text{h.c.}$
2. $\phi^\dagger \phi G^a_{\mu\nu} G^{\mu\nu}_a$
3. $\bar{E}\gamma^\mu P_R E$
4. $\int d^3y \; e^{-(x-y)^2/\sigma^2} \; \phi^\dagger\phi(x)\phi^\dagger\phi(y)$
5. $i(\phi^\dagger\phi)^2$
6. $\phi^\dagger\tilde{\phi}$

2 Gluinos

Suppose that a Majorana fermion is added to the standard model, transforming as an octet under $SU_c(3)$ and a singlet under $SU_L(2)$, that is, $P_L\tilde{G} = (\mathbf{8}, \mathbf{1}, 0)$, or

$$\delta P_L\tilde{G}^\alpha = -f_{\alpha\beta\gamma} w_3^\beta P_L\tilde{G}^\gamma \,, \qquad D_\mu P_L\tilde{G}^\alpha = (\partial_\mu \delta_{\alpha\gamma} + g_3 f_{\alpha\beta\gamma} G^\beta_\mu) P_L\tilde{G}^\gamma \,.$$

Show that the reality of $f_{\alpha\beta\gamma}$ ensures that $P_R\tilde{G}^\alpha$ transforms in the *same* way as $P_L\tilde{G}^\alpha$ (that is, the conjugate of the adjoint representation is the adjoint representation). Verify therefore that the mass term

$$-\frac{m}{2}\overline{\tilde{G}}\tilde{G}$$

is gauge invariant and *can* appear in the Lagrangian. Therefore the $\tilde{G}$ field could possess a mass, independent of the Higgs mechanism (unlike all other spinor fields in the model).

3 Generation Symmetry

Consider the following possible symmetry, which acts on the generation indices of the fields in the Standard Model:

$$\begin{aligned}
L_m &\to \left(P_L U_{G,mn} + P_R U^*_{G,mn}\right) L_n \\
E_m &\to \left(P_L U^*_{G,mn} + P_R U_{G,mn}\right) E_n \\
Q_m &\to \left(P_L U_{G,mn} + P_R U^*_{G,mn}\right) Q_n \\
U_m &\to \left(P_L U^*_{G,mn} + P_R U_{G,mn}\right) U_n \\
D_m &\to \left(P_L U^*_{G,mn} + P_R U_{G,mn}\right) D_n
\end{aligned}$$

where U_G is a 3×3 unitary matrix. The role of U_G is to rotate between the three generations of the model.

Show that this generational symmetry is a symmetry of the kinetic terms in the Lagrangian. But show that it is only a symmetry of the Yukawa interactions if they satisfy a special condition. What is that condition, and do the Yukawa interactions of the standard model actually satisfy it?

Extra credit: modify the standard model by making the Yukawa matrices satisfy the property required for generational symmetry to be a (classical) symmetry of the model. Is U_G or any subgroup of U_G anomaly-free? Could gauge couplings to U_G or any subgroup be consistently added to the model?

4 Triplet Higgs

Suppose that the standard model is supplemented by a second complex Higgs field that transforms as a triplet of $SU_L(2)$ rather than as a doublet; *i.e.*:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

and

$$\delta_2 \psi = i\omega_2^a t_a \psi$$

with

$$t_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; t_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose also that the hypercharge, Y, of the field ψ is zero.

4.1 electromagnetism

First, verify that the matrices t_1, t_2, t_3 satisfy the $SU_L(2)$ commutation relations.

What are the electric charges of the component fields, ψ_1, ψ_2 and ψ_3? Suppose the potential for ψ and the usual Higgs field, ϕ, is minimized when:

$$\phi = \phi_{min} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$\psi = \psi_{min} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(u + iw) \\ 0 \end{pmatrix}.$$

Do these values respect the electromagnetic gauge group $U_{em}(1)$ generated by the electric charge $Q = T_3 + Y$?

4.2 Mass spectrum

Find the masses of the spin-one fields in the mass basis, $W^{\pm}_{\mu}$, Z_μ and A_μ, and find the mixing angle θ relating A_μ, Z_μ to B_μ, W^3_μ via $Z_\mu = W^3_\mu \cos\theta - B_\mu \sin\theta$ and $A_\mu = B_\mu \cos\theta + W^3_\mu \sin\theta$. Is the mass relation $M_W = M_Z \cos\theta$ still valid?

5 Extra U(1)

Suppose the Standard Model is extended to contain an extra $U(1)$ gauge boson F_μ, with gauge coupling g_4. We will refer to the new gauge symmetry as $U_{B-L}(1)$. Suppose that the Higgs boson has charge 0 under this gauge boson, but the left-handed lepton doublet $P_L L$ has charge -1. Take the charge assignments of the standard model fields under $U_{B-L}(1)$ to be the same in each generation.

5.1 Charge assignments

Name the charges of the 5 particles of the standard model under the new $U(1)$ symmetry q_E, $q_L = -1$, q_U, q_D, and q_Q.

Find 3 conditions on these charges from the requirement that the Yukawa couplings respect the $U_{B-L}(1)$ gauge symmetry.

Find 5 additional conditions from the 5 anomaly cancellation conditions,

$$(3,3,1')\,,\quad (2,2,1')\,,\quad (1,1,1')\,,\quad (1,1',1')\,,\quad (1',1',1')\,,$$

where the primes refer to the new field.

Show that *no* choice of q_E, q_U, q_D, q_Q can satisfy all of these conditions. Therefore, the standard model augmented by $U_{B-L}(1)$ is *not* a valid theory.

5.2 Right handed neutrino

Now consider adding, in addition, a field N which is a singlet under $SU(2)$ and $SU(3)$, and which carries hypercharge 0. This is allowed in the standard model because it does not contribute to any of the gauge anomalies.

Write down the possible Lagrangian terms containing the N field which can be added to the standard model. These interactions should include a new Yukawa coupling to the Higgs field. The existence of this Yukawa coupling places a constraint on the value of q_N. Revisit the anomaly calculation of the last subsection and show that the inclusion of the N field makes the $U_{B-L}(1)$ current anomaly free for a specific choice of q_Q *etc*. Therefore the addition of a field N is a requirement for introducing a gauged $B-L$ current in the standard model.

Why did we call the gauge current $B-L$?