Physics 742 Homework 5

1 Crossing symmetry

Use the results in the book for the differential Møller cross-section and crossing symmetry to derive the differential cross-section for Bhabha scattering in the ultrarelativistic limit, but still at $s \ll M_Z^2$:

$$\frac{d\sigma}{du dt}(e^- e^+ \to e^- e^+) = \frac{-2\pi\alpha^2}{s^2}\left(\left|\frac{u}{s} + \frac{u}{t}\right|^2 + \frac{t^2}{s^2} + \frac{s^2}{t^2}\right)\delta(s+t+u)\,.$$

(Hint: this is easy.)

2 Diagram drawing practice

For each proposed process:

- If it can occur at tree level, draw *all* diagrams contributing at tree level.
- If it can occur but only at the loop level, find *one* loop-level diagram giving rise to the process.
- If the rate or cross-section in the standard model is *zero*, write down the symmetry principle which forbids the process.

1. $b\bar{s} \to s\bar{b}$ (involved in B_s meson oscillation)
2. $b\mu^+ \to c\nu_\mu$ scattering
3. $ud \to e^+\nu_e$ scattering
4. The decay $b \to s\gamma$
5. $e^- u \to \nu_e c$
6. $ud \to udh$ Higgs production in up-down scattering

3 Narrow resonances

This problem is easy and will involve little calculating. It illustrates how some of the things we have learned about resonances and initial state radiation apply even for particles about which you have no theoretical understanding.

The $\Upsilon(4s)$ is a narrow resonance, interpreted to be a $b\bar{b}$ bound state. Its mass and width are $m_{\Upsilon(4s)} = 10.580$ GeV, $\Gamma_{\Upsilon(4s)} = 14$ MeV, with a branching fraction to electrons of $\mathrm{Br}(\Upsilon(4s) \to e^+e^-) = 2.8 \times 10^{-5}$. It is experimentally useful because the $\Upsilon(4s)$ decays with almost 100% probability via $\Upsilon(4s) \to B\bar{B}$, with B a meson containing a $\bar{b}$ quark and $\bar{B}$ a meson containing a b quark. This gives a convenient way to produce B meson pairs approximately at rest, which has been exploited by the B-factories, BaBar and Belle.

What is the cross-section for $e^+e^- \to B\bar{B}$ on the $\Upsilon(4s)$ resonance? Hint: the spin of the $\Upsilon(4s)$ is 1. Use Equation (6.4.17).

What, approximately, is the correction to this cross-section formula due to the radiation of soft photons from the e^+ and e^-? Hint: examine the argument leading to Equation (6.7.19).

4 Beta functions

The mass of the b quark, as listed in the book, is 4.24 GeV. In a previous homework assignment, in computing the Higgs decay rate, I advocated using a much smaller value. Let us see why.

The scale dependence of the strong coupling $g_3(\mu)$ and of $h_{33}(\mu)$ (the Yukawa coupling responsible for the bottom mass) are, in the regime $m_b < \mu < m_t$,

$$\frac{\mu\partial}{\partial\mu} g_3(\mu) = \frac{-23}{3}\,\frac{g_3^3(\mu)}{16\pi^2}\,, \tag{1}$$

and

$$\frac{\mu\partial}{\partial\mu} h_{33}(\mu) = -8\,\frac{h_{33}(\mu) g_3^2(\mu)}{16\pi^2}\,. \tag{2}$$

In both cases, I have assumed that the strong coupling is the largest coupling and I have simply ignored terms involving other couplings. I have also ignored higher orders, that is, $g_3^5/(16\pi^2)^2$ and $h_{33}g_3^4/(16\pi^2)^2$ terms.

The value of $g_3(\mu)$ is such that $\alpha_3(\mu = M_Z) = 0.118$. The value of h_{33} is determined by the tree level expression, $m_b = h_{33}v/\sqrt{2}$, **IF** you evaluate $h_{33}(\mu)$ using $\mu = m_b$, that is, $h_{33}(\mu = m_b) = m_b\sqrt{2}/v$.

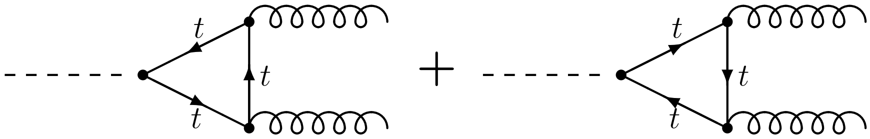
The Higgs decay process involves the $Hb\bar{b}$ vertex at a large energy scale $\simeq M_H$. Therefore, it is $h_{33}(\mu = M_H)$ which is relevant in evaluating the Higgs decay width.

Evaluate $h_{33}(\mu = M_H)$, assuming $M_H = 126$ GeV. You should be able to solve for $h_{33}(\mu = M_H)$ *explicitly* (that is, analytically without resort to numerical methods). However, if you get frustrated, solve the differential equations by numerical means.

Using the tree level relation $m_b = h_{33}\sqrt{2}/v$, evaluate what b mass you should use, to evaluate the Higgs decay width.

5 Higgs to Gluon Gluon

There is no Lagrangian interaction between a Higgs boson and two gluons, but one is induced by loops. The dominant diagrams are:



5.1 Only top quarks

Explain why the large top quark mass allows us to neglect the contribution of all other quarks in the loop.

5.2 Operator analysis

Explain why the lowest-order gauge invariant, CP invariant, scalar operator by which two gluons can couple to a Higgs field is $G^{\alpha}_{\mu\nu} G^{\mu\nu}_{\alpha} \phi^{\dagger}\phi$. Therefore the diagrams above should converge in the UV by $6 - 4 = 2$ powers of the loop momentum. On the other hand, since there are 3 external lines which are each dimension-1 fields, we expect *naively* linearly divergent behavior in any diagram giving rise to this coupling.

5.3 Matrix element

Consider the top-loop diagrams shown above. Write an expression for each diagram, labeling the incoming Higgs momentum P, the outgoing gluon momenta P_1 and $P_2 = P - P_1$, the external gluon gauge indices μ, ν, and the loop momentum Q. Remember to include the top quark mass in the propagators. Argue that the integral is naively linearly large Q divergent, but that after taking the Dirac traces, the UV behavior at large Q is *at worst* logarithmically divergent.

5.4 Gauge invariance and finiteness

In fact, our operator dimension argument shows that the integrals should converge as $\int d^4Q/Q^6$ at large Q. And because the loop should reproduce field strengths $G_\alpha^{\mu\nu}$, we expect that the loop should vanish if either P_1 or P_2 is exactly zero.

To see that this behavior actually occurs, consider the diagrams in the simplifying case $P, P_1 = 0$, which also captures the dominant large-Q behavior. Perform the numerator traces in $D = 4 - 2\epsilon$ dimensions, and show that the sum of the diagrams reduces to

$$\mathcal{M}(P=0, P_1=0) = -i\frac{4m_t^2 g_3^2}{v}\delta^{\alpha\beta}\epsilon_\mu\epsilon'_\nu \int \frac{d^DQ}{(2\pi)^D}\frac{(Q^2+m_t^2)\eta^{\mu\nu} - 4Q^\mu Q^\nu}{(Q^2+m_t^2 - i\epsilon)^3} \equiv \delta^{\alpha\beta}\epsilon_\mu\epsilon'_\nu\mathcal{M}^{\mu\nu}\,, \tag{3}$$

where $\delta^{\alpha\beta}$ contracts the color indices on the gluons, ϵ and ϵ' are the polarization vectors of the gluons, and m_t/v is the Higgs Yukawa coupling to a top quark. Argue that $\mathcal{M}^{\mu\nu}$ must be proportional to $\eta^{\mu\nu}$: $\mathcal{M}^{\mu\nu} = M\eta^{\mu\nu}$. Therefore,

$$\eta_{\mu\nu}\mathcal{M}^{\mu\nu} = DM = -i\frac{4m_t^2 g_3^2}{v}\int \frac{d^DQ}{(2\pi)^D}\frac{(D-4)Q^2 + Dm_t^2}{(Q^2+m_t^2 - i\epsilon)^3}\,. \tag{4}$$

Evaluate this expression in $D = 4 - 2\epsilon$ dimensions, with $\epsilon > 0$, and show that it vanishes. Therefore the loop integral at vanishing P, P_1 vanishes, and the behavior at large Q will be softer than d^4Q/Q^4.

Also check gauge invariance by replacing ϵ^μ with P_1^μ in the expression for $\mathcal{M}$ (before expanding in small P). Show, using the shift symmetry of the integrand, that the diagram gives exactly zero (for any values of P_1, P_2).

5.5 Extra credit: evaluation

Assume that the loop momentum $Q \sim m_t$, and treat $m_t \gg m_H$, which allows one to expand in $P_1, P_2 \ll Q, m_t$. Evaluate $\mathcal{M}^{\mu\nu}$ explicitly to second order in P_1, P_2. This can be done, for instance, by Taylor expanding in P. The previous section shows that the zero-order term vanishes on integration. Show that the linear term vanishes and that the quadratic term is

$$\mathcal{M}^{\mu\nu} = \frac{\alpha_3}{3\pi v}\left(\eta^{\mu\nu}P_1\cdot P_2 - P_2^\mu P_1^\nu\right). \tag{5}$$

Confirm that this result is transverse in P_1 and P_2.

Hint: show that the only possible tensor structure which is second-order and transverse in P_1 and P_2 is $(\eta^{\mu\nu}P_1\cdot P_2 - P_2^\mu P_1^\nu)$. Find the coefficient by evaluating $\eta_{\mu\nu}M^{\mu\nu}$.

5.6 Partial width

Use the result found above to show that the partial width of a Higgs boson to decay to a gluon pair is $\Gamma_{H\to gg} = \frac{m_H^3\alpha_3^2}{72\pi^3 v^2}$. Compare the decay width to the decay width to $b\bar{b}$.

You should find that the partial decay width is significantly smaller than the decay width to the $b\bar{b}$ final state. Therefore this decay channel is not very important. But will see that the inverse process, $gg \to H$, dominates Higgs boson production in hadron colliders. Similarly, loops involving t and W allow the decay $H \to \gamma\gamma$, which has a small branching fraction but a clean signature and which was the other process by which the Higgs boson was discovered.