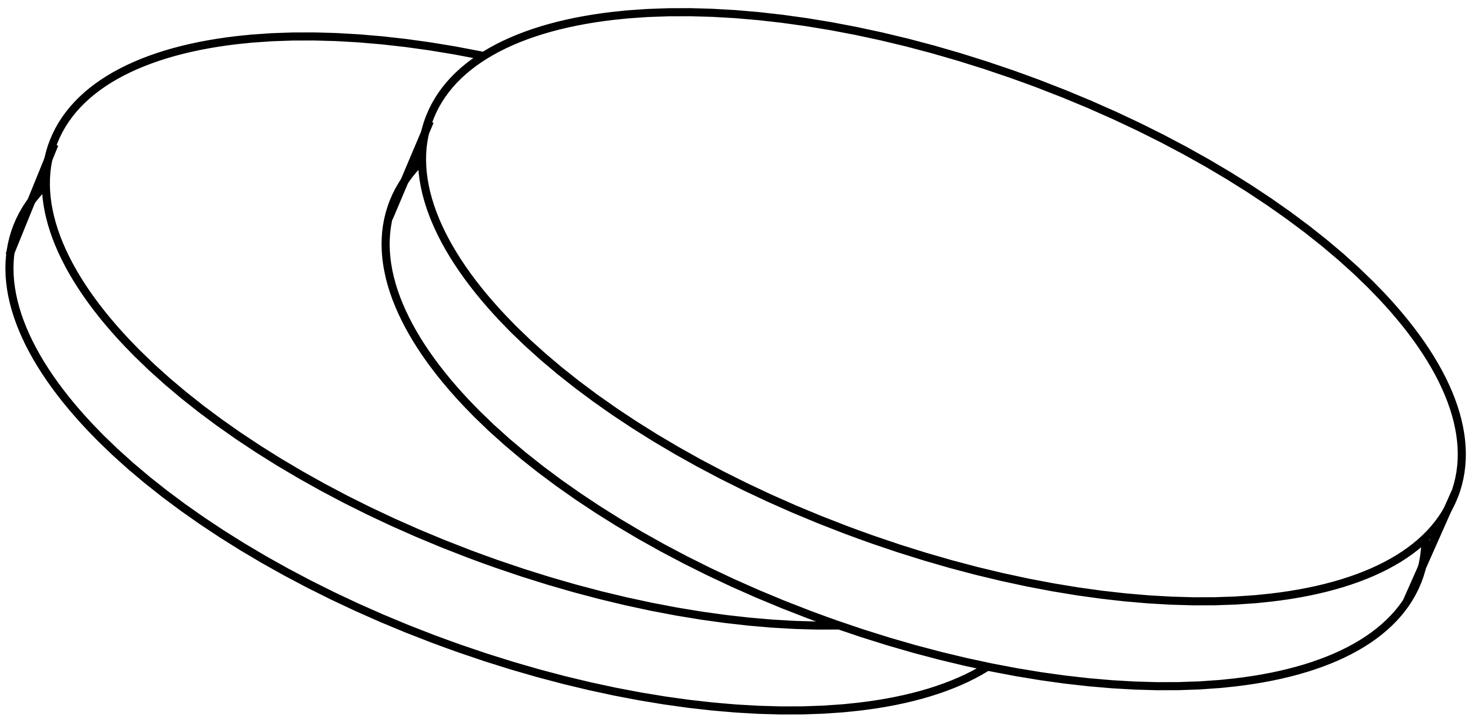
Second Order Hydro in QCD

Guy D. Moore, Mark Abraao York

- Why do hydrodynamics in QCD?
- Why find 2'nd order coefficients and what are they?
- Kinetic theory: setup
- Kinetic theory: details
- Interesting physics along the way
- Conclusions

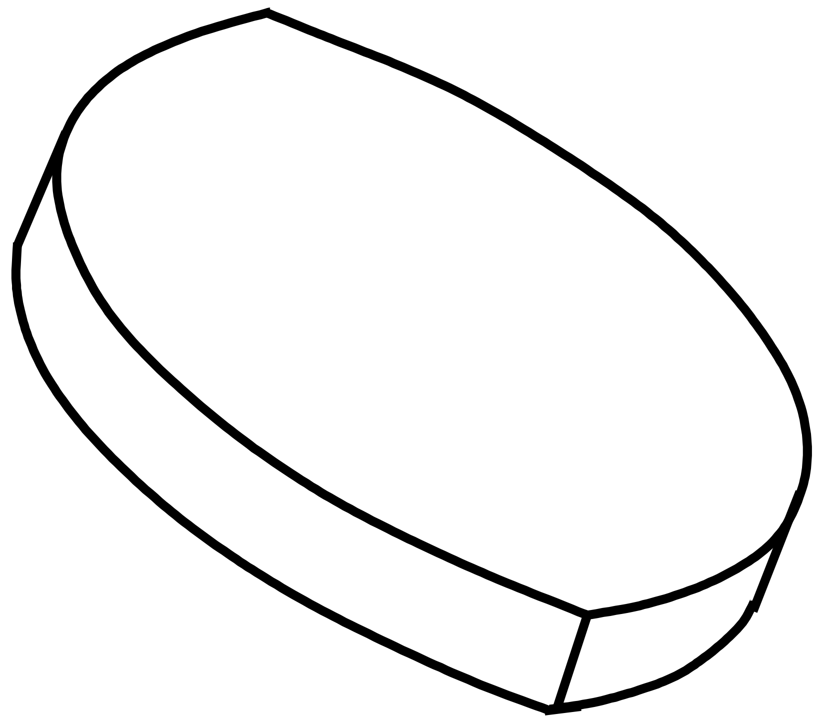
Heavy ion collisions

Accelerate two heavy nuclei to high energy, slam together.



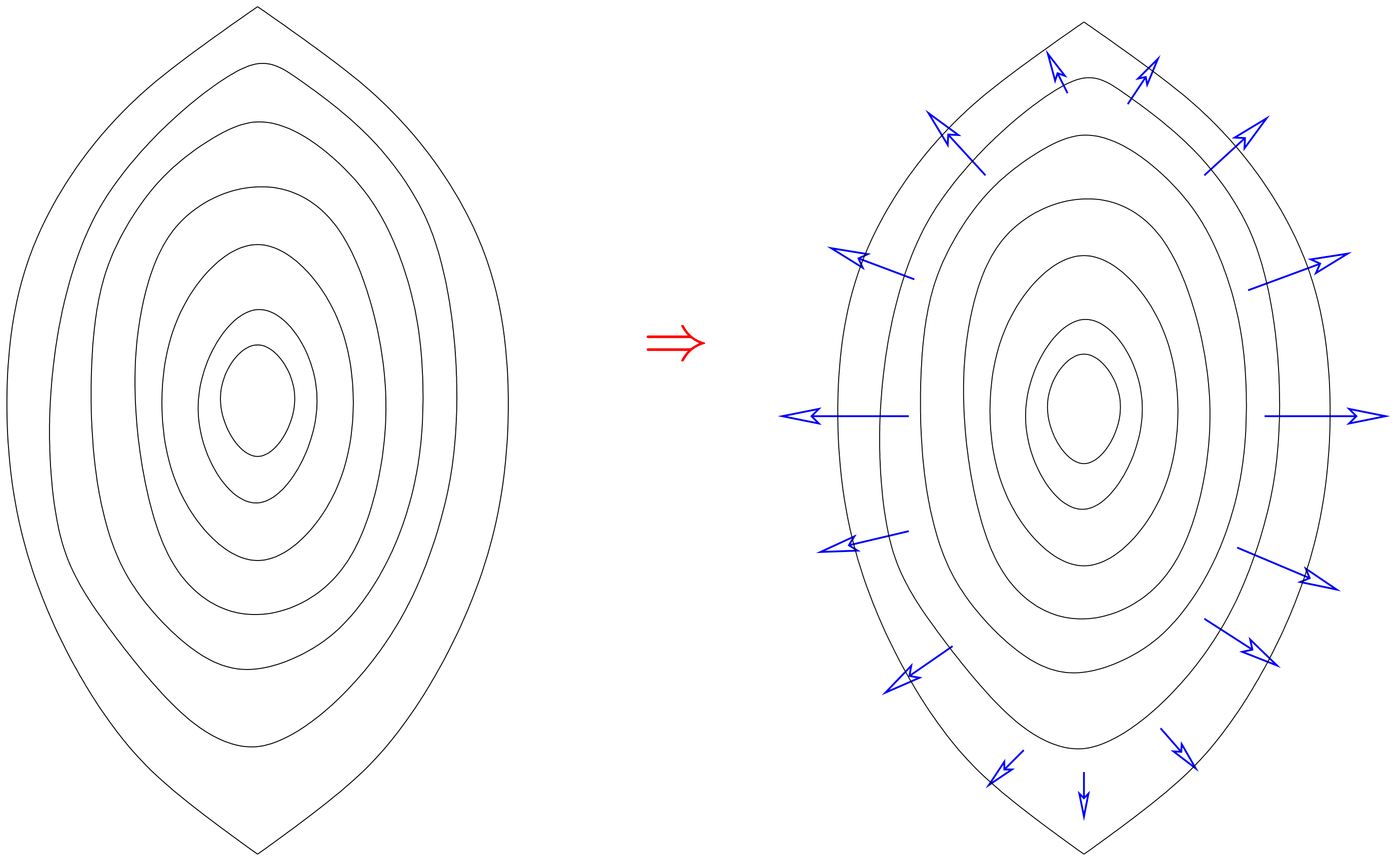
Just before: Lorentz contracted nuclei

After the scattering: region where nuclei overlapped:
"Flat almond" shaped region of $q, \bar{q}, g$ which scattered.



$\sim$2 thousand random $\mathbf{v}$ quarks+gluons: isotropic in xy plane

local CM motions

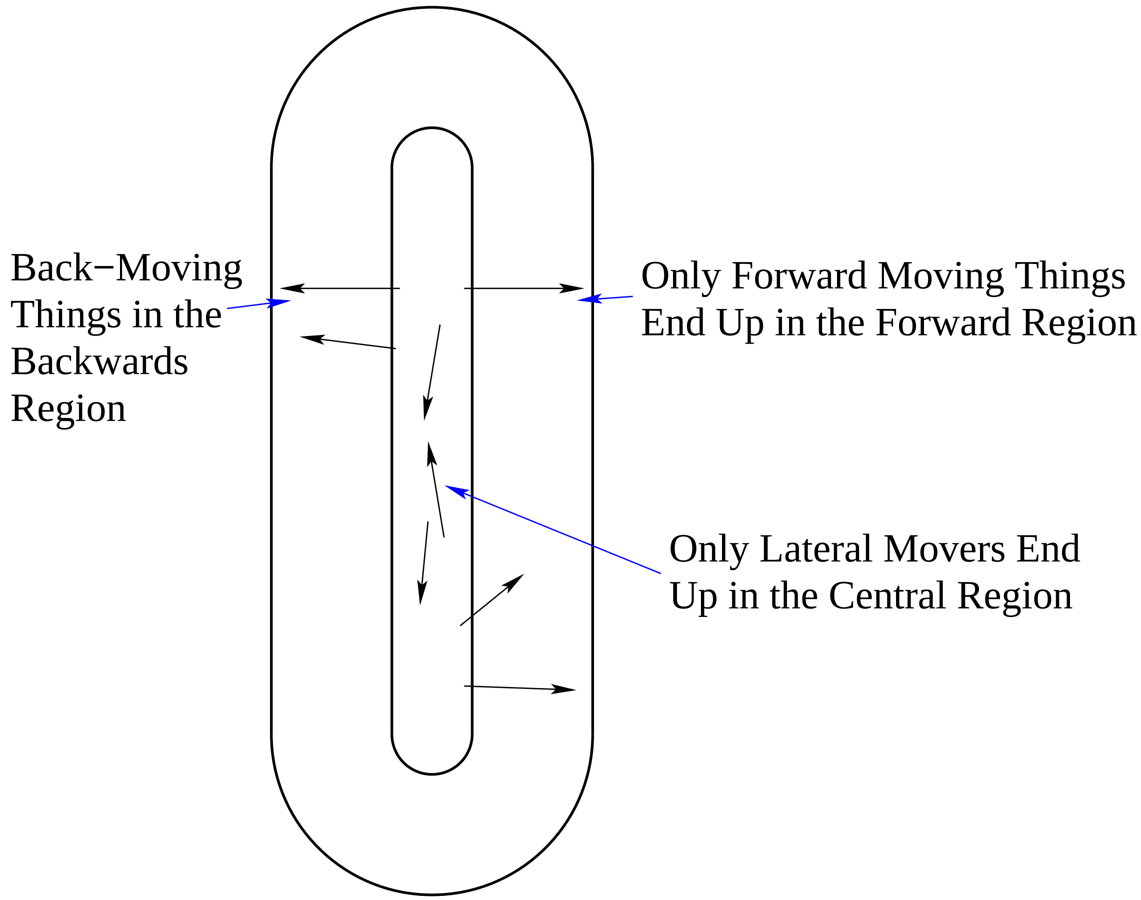


Pressure contours Expansion pattern

Anisotropy leads to anisotropic (local CM motion) flow.

Momentum Selection

Side-on view of the flat almond as it expands



Space aniso. $\longrightarrow$ aniso. of "particle" p distrib.

Free particle propagation:

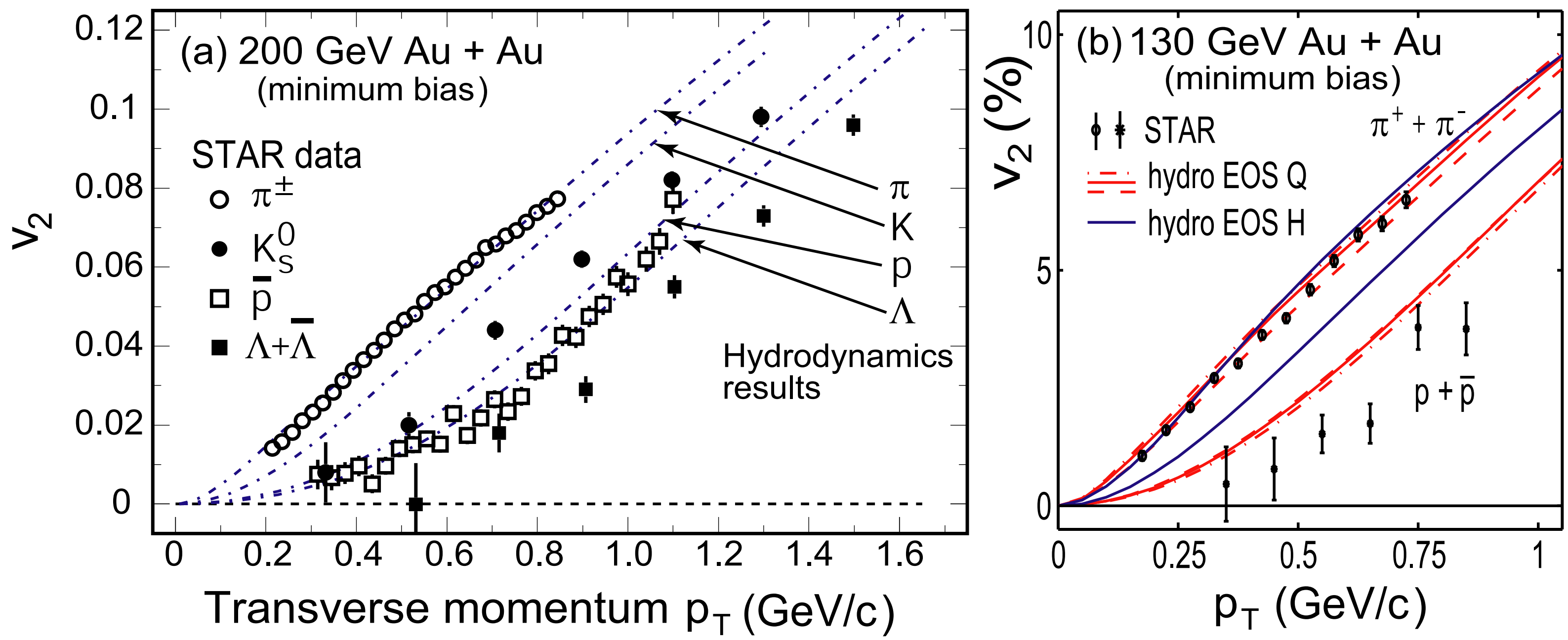
- Particle distributions locally triaxial, $\langle v_x^2 \rangle < \langle v_y^2 \rangle$, but
- System-average CM flow velocities $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Total particle distribution $\langle v_x^2 \rangle = \langle v_y^2 \rangle$

Efficient scattering:

- Drives system *locally* towards $\langle v_{x,\mathrm{relative}}^2 \rangle = \langle v_{y,\mathrm{relative}}^2 \rangle$
- System-average CM flow still has $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Adding these together, $\langle v_{x,\mathrm{tot}}^2 \rangle > \langle v_{y,\mathrm{tot}}^2 \rangle$

Net "Elliptic Flow" $v_2 \equiv \frac{v_x^2 - v_y^2}{v_x^2 + v_y^2}$ measures scattering

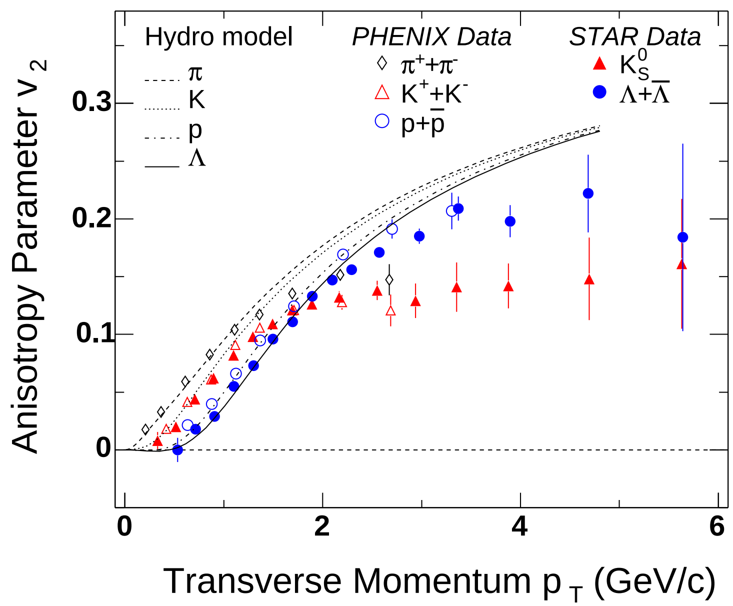
Elliptic flow is measured

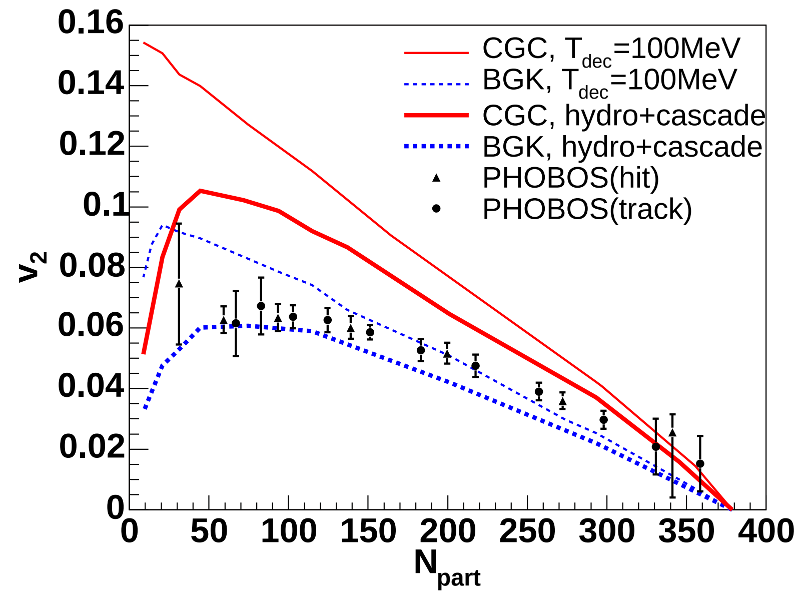


STAR experiment, minimum bias... .

We should try to understand it theoretically.

First attempt: ideal hydro





Works, DEPENDING on initial conditions.

Corrections to ideality exist, but are “small” (?)

Can we quantify that?

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

Ideal hydro works well: corrections *eg, viscosity* small
Claim: "Most Perfect Liquid" exotic behavior. **Quantify!**

Nonideal Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial, \epsilon, u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial\mu, \partial\epsilon) + \mathcal{O}(\partial^2\mu, (\partial\mu)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

Viscous hydro

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \quad ?$$

Because in **relativisitc** setting, it is

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- Unstable: $v > c$ prop + non-uniform flow velocity $\rightarrow$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem only on short length scales where $\eta|\sigma| \sim P$. But numerics must treat these scales (or "numerical viscosity" which exceeds η is present)

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_{\Pi}\, u^{\alpha}\partial_{\alpha}\sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \to -\Pi$ in order-2 term:

$$\tau_{\Pi}\, u^{\alpha}\partial_{\alpha}\Pi^{\mu\nu} \equiv \tau_{\Pi}\, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_{Π})

But why only one 2'nd order term???

Second order hydrodynamics

It is more consistent to include all possible 2'nd order terms.
Assume *conformality* and *vanishing chem. potentials*:
5 possible terms Baier *et al*, [arXiv:0712.2451]

$$\begin{aligned}\Pi^{\mu\nu}_{2\text{ ord.}} = \; & \eta\tau_\Pi \left[u^\alpha\partial_\alpha\sigma^{\mu\nu} + \frac{1}{3}\sigma^{\mu\nu}\partial_\alpha u^\alpha\right] + \lambda_1\left[\sigma^\mu_\alpha\sigma^{\nu\alpha} - (\text{trace})\right] \\ & + \lambda_2\left[\frac{1}{2}(\sigma^\mu_\alpha\Omega^{\nu\alpha} + \sigma^\nu_\alpha\Omega^{\mu\alpha}) - (\text{trace})\right] \\ & + \lambda_3\left[\Omega^\mu{}_\alpha\Omega^{\nu\alpha} - (\text{trace})\right] + \kappa\left(R^{\mu\nu} - \ldots\right) \; , \\ \Omega_{\mu\nu} \equiv \; & \frac{1}{2}\Delta_{\mu\alpha}\Delta_{\nu\beta}(\partial^\alpha u^\beta - \partial^\beta u^\alpha) \;\; [\text{vorticity}] \, .\end{aligned}$$

Now, besides η, we have 5 more unknown coefficients.

Second order: philosophy

(nonideal) hydro only consistent if η makes small corr.

These τ_Π, $\lambda_{1,2,3}$ make smaller corrections. κ irrelevant.

Make reasonable estimate for τ_Π, λ_{123}, test sensitivity

Guy argues: Ratios should be relatively robust

- Forget $\frac{\eta}{s}$. Think of $\frac{\eta}{P+\epsilon} = t_\eta$ a *timescale*. Pert: $1/g^4 T$
- Next order: $\frac{\lambda_1}{P+\epsilon} = t_\lambda^2$, $\frac{\eta \tau_\Pi}{P_\epsilon} = t_\pi^2$ Pert: $1/g^8 T^2$
- All are thermalization times. One expects $t_\lambda \sim t_\eta \sim t_\pi$.

Determine ratios where you can, use as priors in fit

Two toy models of QCD

To date, coeff's computed in *toy model* for QCD:
$\mathcal{N}$=4 SYM theory at $N_c, g^2 N_c \to \infty$
(conformal, vast number DOF, many scalars, infinite coupling,...) Baier et al [arXiv:0712.2451], Tata group, [arXiv:0712.2456]

I know another *toy model* for QCD:
Weakly coupled $N_c = 3$, $N_f = 0, \ldots 6$ QCD in pert. theory!
(asymptotically free, mass gap, right number DOF, finite coupling...)

Leading order calculation: theory conformal, same coeff's
Toolkit for calculation: kinetic theory (valid at leading order)

Kinetic theory

Weak coupling: IR-safe corr. funcs nearly Gaussian.
Adequate description in terms of 2-point function.

Value of 2-pt function has interpretation as particle number:
$\phi^\dagger \phi$ is $\frac{1}{2} + \hat{N}$ number operator of free thy.

Leading-order: free propagation. Scatterings "rare".

Allows extra approximation: $\Delta x \sim 1/p \sim 1/T$ small compared to free path $\lambda \sim 1/g^2 T$. Propagation classical, $[x,p] \simeq 0$ "classical phase space" behavior.

Kinetic theory

State, all measurables described by particle distrib. $f_a(x,p)$:

$$T^{\mu\nu}(x) = \sum_a \int_p 2P^\mu P^\nu f_a(x,p)\,, \qquad \int_p \equiv \int \frac{d^3p}{(2\pi)^3 2p^0}$$

(Assumes weak coupling, slow x^μ dependence, little else)

Dynamics: Boltzmann equation (Schwinger-Dyson eq):

$$2P^\mu \partial_{\mu_x} f(x,p) \;\; = \;\; -\mathcal{C}[p, f(x,q)]$$

LHS: particle propagation. $p^0 \equiv \sqrt{\mathbf{p}^2} \equiv p$

RHS: scattering (Im self-energy). Local in x but not p.

Theory dependence all contained in detailed form of $\mathcal{C}[f]$.

In our case, described in detail in AMY5: `hep-ph:0209353`

Two gradient expansions

Hydrodynamics description relies on

$$(t_{\rm therm}, v t_{\rm therm}) \ll (t_{\rm vary}, l_{\rm vary})$$

slow variation in time,space. Expandable order-by-order.

Kinetic theory description relies on

$$(t_{\rm deBroglie} \sim T^{-1}, \lambda_{\rm deBroglie}) \ll (\Gamma^{-1}, \lambda_{\rm mfp})$$

where $\Gamma^{-1} \leq t_{\rm therm}$ is inverse scatt rate.
We don't know how to do this expansion order-by-order!
g^2 corrections lie outside kinetic description.

Expansion in (hydro) gradients

$$f(x,p) = f_0(\beta(x), u(x), p) + f_1(\partial, \beta, u, p) + f_2(\partial^2, \beta, u, p)$$

Subscript counts order in derivatives. β, u and $\epsilon, \vec{P}$ dual

LHS of Boltzmann has 1 deriv: RHS has 0.

$$\mathcal{O}\ 0\text{:} \quad \mathcal{C}[p, f_0(x,q)] = 0 \quad \rightarrow \quad f_{0,a} = \frac{1}{\exp(-\beta u^\mu P_\mu) \pm 1}$$

$$\mathcal{O}\ 1\text{:} \quad 2P^\mu \partial_{\mu_x} f_0(\beta(x), u(x), p) = -\mathcal{C}_1[p, f(q)]$$

where $\mathcal{C}_1$ is $\mathcal{C}$ expanded to lin. order in f_1.

$$\mathcal{O}\ 2\text{:} \quad 2P^\mu \partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

with $\mathcal{C}_{11}$ 2 order in f_1, $\mathcal{C}_2$ lin. order in f_2

First order in expansion

Organize it as

$$f_1(q) = -\mathcal{C}^{-1}_{1,qp} 2P^\mu \partial_\mu f_0(-\beta u^\nu P_\nu)$$

Gradients of free-theory distribution act as source for f_1.

$$2P^\mu \partial_\mu f_0 = -f_0' \beta P^\mu P^\nu (\partial_\mu u_\nu + u_\mu \partial_\nu \beta)$$

Organize source in spherical harmonics. $\ell = 0, 1$ determine u, β:

$$\partial_t \beta = \frac{\beta}{3} \partial_i u_i \qquad \text{and} \qquad \partial_t u_i = \frac{1}{\beta} \partial_i \beta$$

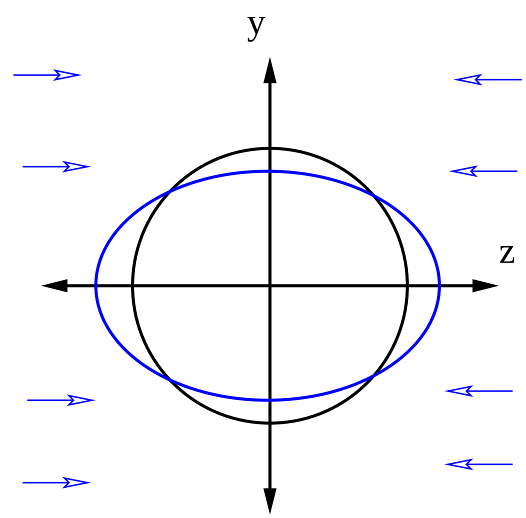
Remaining term is nontrivial:

$$f_1(q) = \mathcal{C}^{-1}_{1,qp} (p_i p_j - p^2 \delta_{ij}/3) \beta \sigma_{ij} f_0'$$

Solution always of form $f_1(p) = \frac{\beta^3}{2} \sigma_{ij} (p_i p_j - p^2 \frac{\delta_{ij}}{3}) \chi(-\beta u_\mu P^\mu)$

Physics so far: viscosity

Consider system under compression: momentum distribution becomes prolate spheroidal (limited by scattering)



Viscosity measures $T_{\mu\nu}$ of this distortion,
so $\eta/P = 4\eta/(P+\epsilon)$ measures extent of prolateness.

Prolateness can differ at different $|p|$;
$\chi(\beta p)$ tells how this varies, η gives some average.

Second order Boltzmann Equation

$$2P^\mu \partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

Organize it as

$$f_2 = -\mathcal{C}_1^{-1}\Big(2P^\mu \partial_\mu f_1 + \mathcal{C}_{11}[f_1]\Big)$$

Term on right acts like a source for 2'nd order departure f_2.
Two pieces: effect of inhomogeneity on 1'st order departure, nonlinearity of collision operator in departure from equilib.

Determining Π^{ij} only requires $\ell = 2$ moment of f_2, which simplifies calculation: only need $\ell = 2$ of RHS.

Term $2P^\mu \partial_\mu f_1$

Inhomogeneous flow when f already skewed. Consider

$$2P^\mu \partial_\mu \sigma_{\alpha\beta}(P^\alpha P^\beta - g^{\alpha\beta} p^2/3)\beta^3 \chi(-\beta u^\gamma P_\gamma)$$

Two types of terms: ∂_μ acts on $\sigma_{\alpha\beta}\beta^3$, or on χ (Dirk: NOTE)

First term:

$$P^\mu P^\nu P^\alpha \left(\partial_\mu \sigma_{nu\alpha} + 3\sigma_{\nu\alpha}\partial_\mu \ln\beta\right)\beta^3\chi$$

Only contributions to Π^{ij} when 2 P's space, one time.

$$Ep_i p_j \left(\partial_0 \sigma_{ij} + 2\partial_i \sigma_{0j} + 3\sigma_{ij}\partial_0 \ln\beta\right)$$

Contributes to τ_Π, λ_2, λ_1. Second term: contributes to λ_1.

What we get so far

One contribution to τ_Π, λ_2, and λ_1.
Automatically in ratio $1 : -2 : -1$. Therefore

$$\lambda_2 = -2\eta\tau_\Pi$$

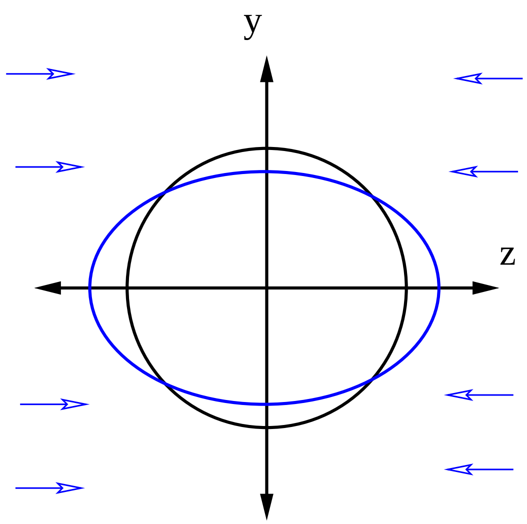
Extra independent (positive) contribution to λ_1.

Detailed values depend on functional form of $\chi(\beta E)$.
Specific *Ansatz* (Grad 14-moment) gives specific values:

$$\frac{\eta\tau_\Pi}{\epsilon + P} = \frac{6\zeta(4)\zeta(6)}{\zeta^2(5)} \left(\frac{\eta}{\epsilon + P}\right)^2 , \quad \frac{\lambda_1}{\eta\tau_\Pi} = 1 \,.$$

But we *solve* for $\chi(\beta E)$–slightly different value

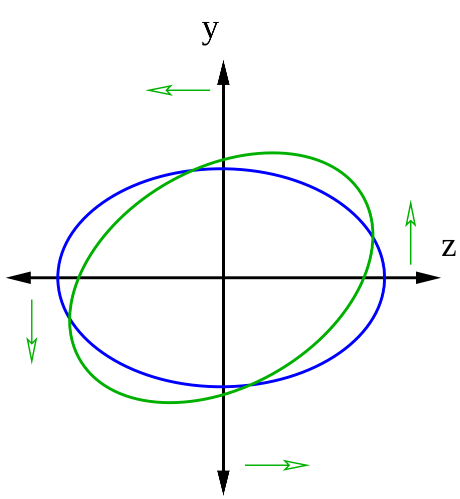
What do these coefficients mean?



Contraction $\rightarrow$ prolateness.

τ_Π: how long prolateness lasts after contraction ends.

λ_1: induced prolateness depends on how prolate it already is!



(Vorticity) rotation

Vorticity means rotation.

λ_2: how much prolateness gets rotated in a rotating system.

Depends on how long prolateness lives.

Hence relation $\lambda_2 = -2\eta\tau_\Pi$.

Hard part: $\mathcal{C}_{11}$

Nonlinear term in collision operator:

"extra" particles scattering from other "extra" particles

Theory dependent. Consider $2 \leftrightarrow 2$ scattering (present in most theories):

$$\mathcal{C}[p, f[q]] \;=\; \int_{kp'k'} (2\pi)^4 \delta^4(P{+}K{-}P'{-}K')|\mathcal{M}^2| \times$$
$$\Big(f(p)f(k)[1{\pm}f(p')][1{\pm}f(k')] - f(p')f(k')[1{\pm}f(p)][1{\pm}f(k)] \Big)$$

First order expansion: define $\bar{f}_1 = f_0(1{\pm}f_0)f_1$.

$$\Big(f(p)f(k)[1{\pm}f(p')][1{\pm}f(k')] - f(p')f(k')[1{\pm}f(p)][1{\pm}f(k)] \Big)$$
$$= \quad 0 + f_0(p)f_0(k)[1{\pm}f_0(p')][1{\pm}f_0(k')]\Big(\bar{f}_1(p){+}\bar{f}_1(k){-}\bar{f}_1(p'){-}\bar{f}_1(k') \Big)$$

That's what we needed in defining $\mathcal{C}_1$. Used twice already!

Next order: $f_0(p)f_0(k)[1\pm f_0(p')][1\pm f_0(k')]$ times

$$
\begin{aligned}
&\bar{f}_1(p)\bar{f}_1(k)f_0(p)f_0(k)(e^{\frac{p+k}{T}}-1)+\bar{f}_1(p')\bar{f}_1(k')f_0(p')f_0(k')(1-e^{\frac{p+k}{T}})\\
&+\Big[\bar{f}_1(p)\bar{f}_1(p')f_0(p)f_0(p')\left(e^{\frac{p}{T}}-e^{\frac{p'}{T}}\right)+(p'\to k')\\
&\quad+(p\to k)+(p,p'\to k,k')\Big]
\end{aligned}
$$

Note that $\bar{f}_1(p)\propto\sigma_{ij}(p_ip_j-\delta_{ij}p^2/3)$.
In evaluating $\langle S_{ij}|\mathcal{C}_{11}[f_1]\rangle$ we meet angular integrations:

$$
\begin{aligned}
&\text{defining } p_{\langle i}q_{j\rangle}=\frac{3p_iq_j+3q_ip_j-2p\cdot q\delta_{ij}}{6}\,,x_{pq}\equiv p\cdot q\,,\\
&\sigma_{lm}\sigma_{rs}\int d\Omega_{\rm global}p_{\langle i}p_{j\rangle}q_{\langle l}q_{m\rangle}r_{\langle r}r_{s\rangle}\\
=\;&\frac{4}{35}\left(\sigma_{il}\sigma_{jl}-\frac{\delta_{ij}}{3}\sigma_{lm}\sigma_{lm}\right)\\
&\times\Big(3x_{pq}x_{pr}x_{qr}-x_{pp}x_{qr}^2-x_{qq}x_{pr}^2-x_{rr}x_{pq}^2+2x_{pp}x_{qq}x_{rr}/3\Big)
\end{aligned}
$$

Using these, one can bludgeon $\mathcal{C}_{11}$ term to death. Contributes only to λ_1.

Subtlety!

Preceding assumed that matrix element $|\mathcal{M}^2|$ is f independent.
In gauge and Yukawa theories, f enters $\mathcal{M}$ through screening!
Change in $f_0 \to f_0 + f_1$ changes screening, leading to correction to $|\mathcal{M}|^2$ linear in f_1.

This is where things get hard.

So I won't tell you about it except one detail.

Antiscreening and plasma instability

Plasmas screen all interactions $except$ $\omega \to 0$ magnetic.
Deep consequence of dimensional reduction: in 3D theory, rotation+gauge inv mean F^2, not A^2 op can appear.

Nonzero f_1: plasma is not rotation non-invariant. Former (equilibrium!) argument fails: magnetic "mass" possible. But $\int_\Omega$ of screening effect is zero!

Angular average zero, value in some directions nonzero $\longrightarrow$ Screening is negative in some directions (plasma instabilities)

Plasma instability for us

We are working perturbatively – won't see full instability.

$$G^{\mu\nu} = \frac{1}{G_0^{-1} - \delta\Pi} \simeq G_0^{\mu\nu} + G_0^{\mu\alpha}\delta\Pi_{\alpha\beta}G_0^{\beta\nu}$$

Now $\delta\Pi$ doesn't vanish but $G_0 \sim 1/q^2$:

$$G^{\mu\nu}(q) \sim \frac{1}{q^2} + \delta\Pi\frac{1}{q^4}$$

more IR singular. Potential for log IR singularity.

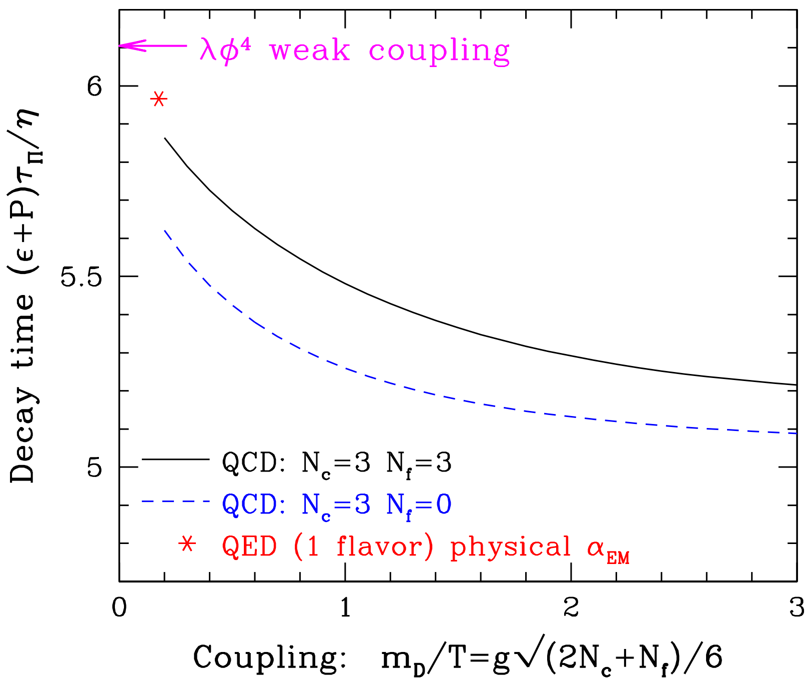
Turns out to cancel in angular stuff for $2 \leftrightarrow 2$,

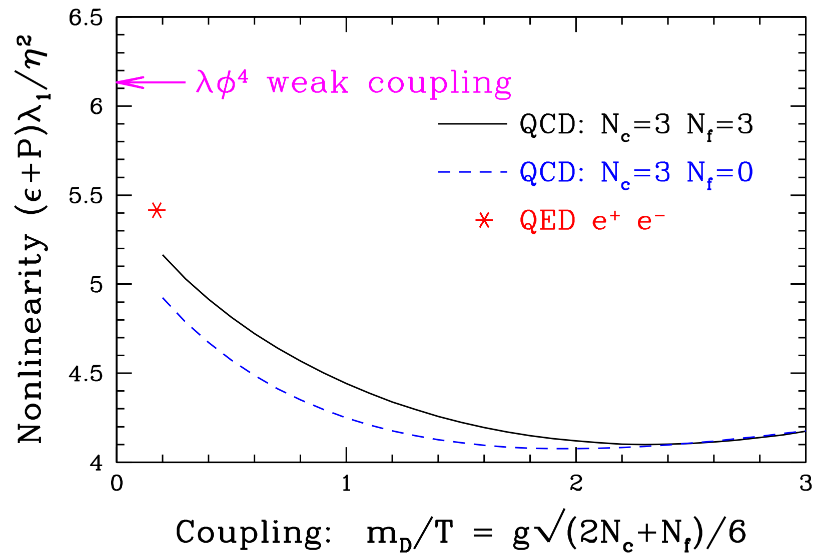
but not for $1 \leftrightarrow 2$ (splitting).

Splitting rate is IR log singular. I don't have exact results!

Results

$\lambda_3 = \kappa = 0$. $\lambda_2 = -2\eta\tau_\Pi$. τ_Π, λ_1 nontrivial:





Size of uncertainty is thinner than lines in plots!

Ratios are very stable with value of coupling.

QCD vs SYM comparison

Ratio	QCD value	SYM value
$\frac{\tau_\Pi(\epsilon+P)}{\eta}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	4.1 to 5.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	0	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Good news: Not qualitatively different.

Bad news: "exact" kinetic theory relation $\lambda_2 = -2\eta\tau_\Pi$ not actually general.

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Shear viscosity should be quantified!
- Requires expansion to 2'nd order in gradients
- Calculation in pert. QCD is intricate.
- Ratios are relatively robust. But Pert Thy and SYM give rather different predictions in detail.

Limitation of kinetic theory method?

How does one compute non-kinetic corrections?