Second-Order Relativistic Hydrodynamics

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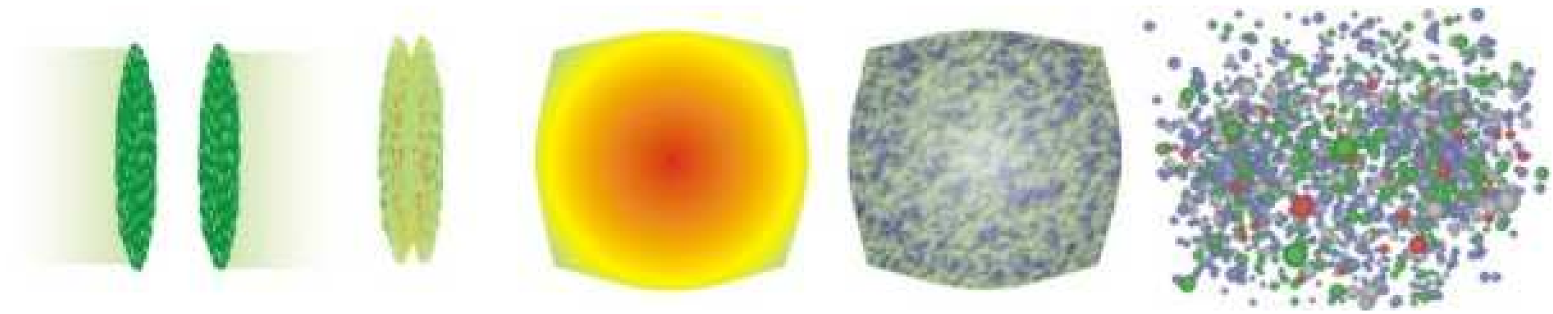
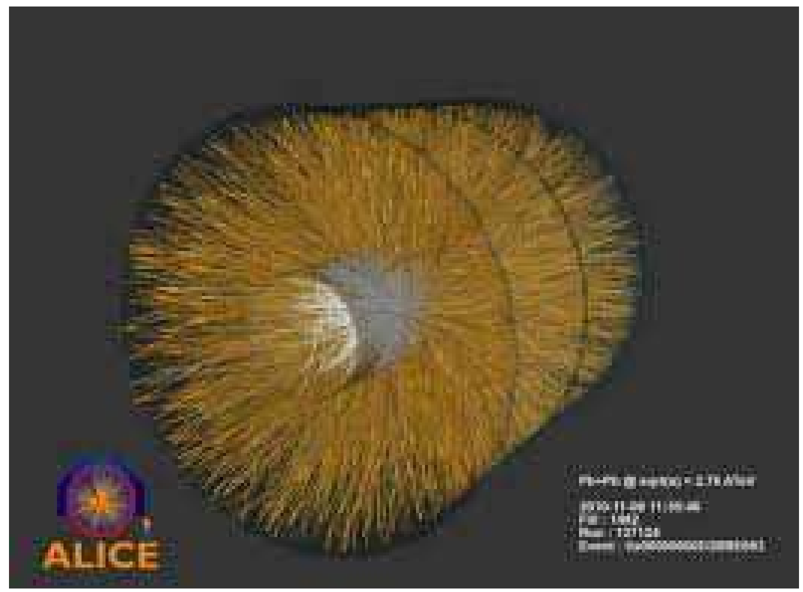
- Why do we want to do relativistic Hydro?
- Why second order hydro, and what are coefficients?
- Perturbative results and limitations
- Kubo Relations for Coefficients
- Self-consistency: hydro's contrib. to hydro coeff.
- Conclusions

LHC collides lead nuclei ($82p + 126n = 208$ nucleons)

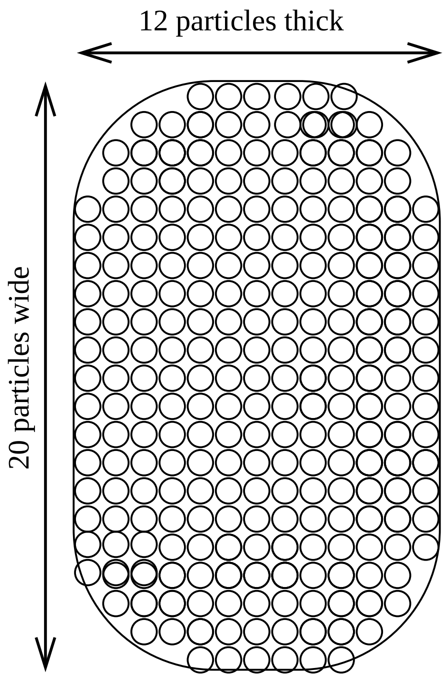


leading to 3200 charged,
> 1600 neutral particles
between $\theta = 40^\circ$ and
$\theta = 140^\circ$ $(-1 < \eta < 1)$



Each n, p gets "torn open," spilling out many $g, q, \bar{q}$ inside

Hot ball of 5000 excitations

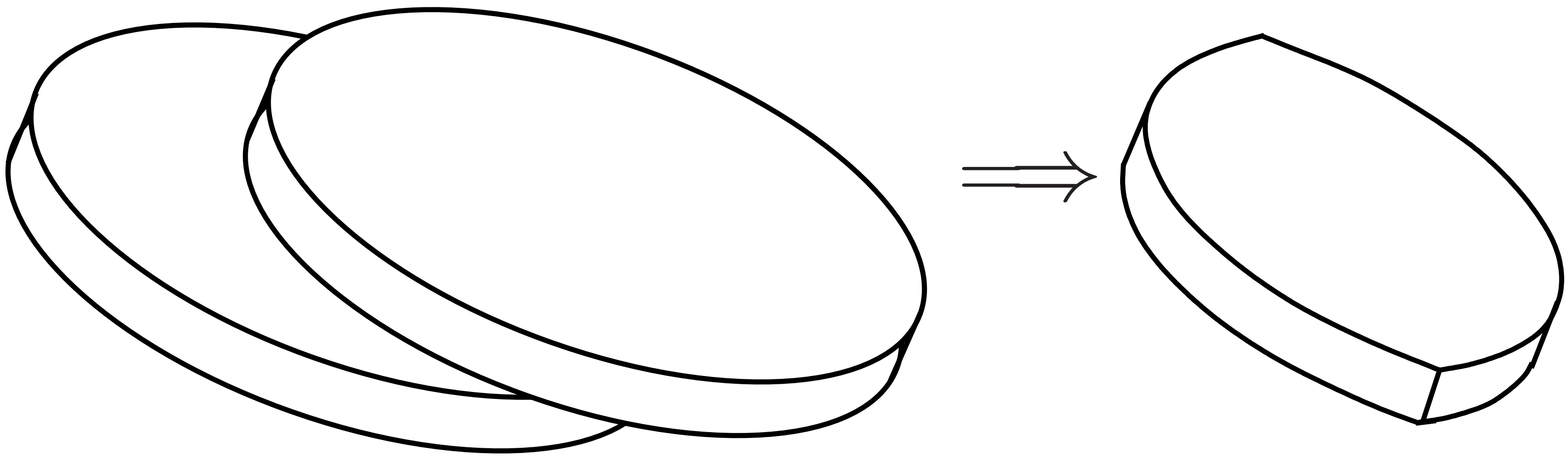


5000 excitations is around $20 \times 20 \times 12$ across. Enough to show collective or "fluid" behavior?

Hydrodynamics: Many "subsystems" big enough for *local* equilibration in each (Different regions with different $T, \vec{v}$,...). Not obvious but plausible

Testing for local equilibration

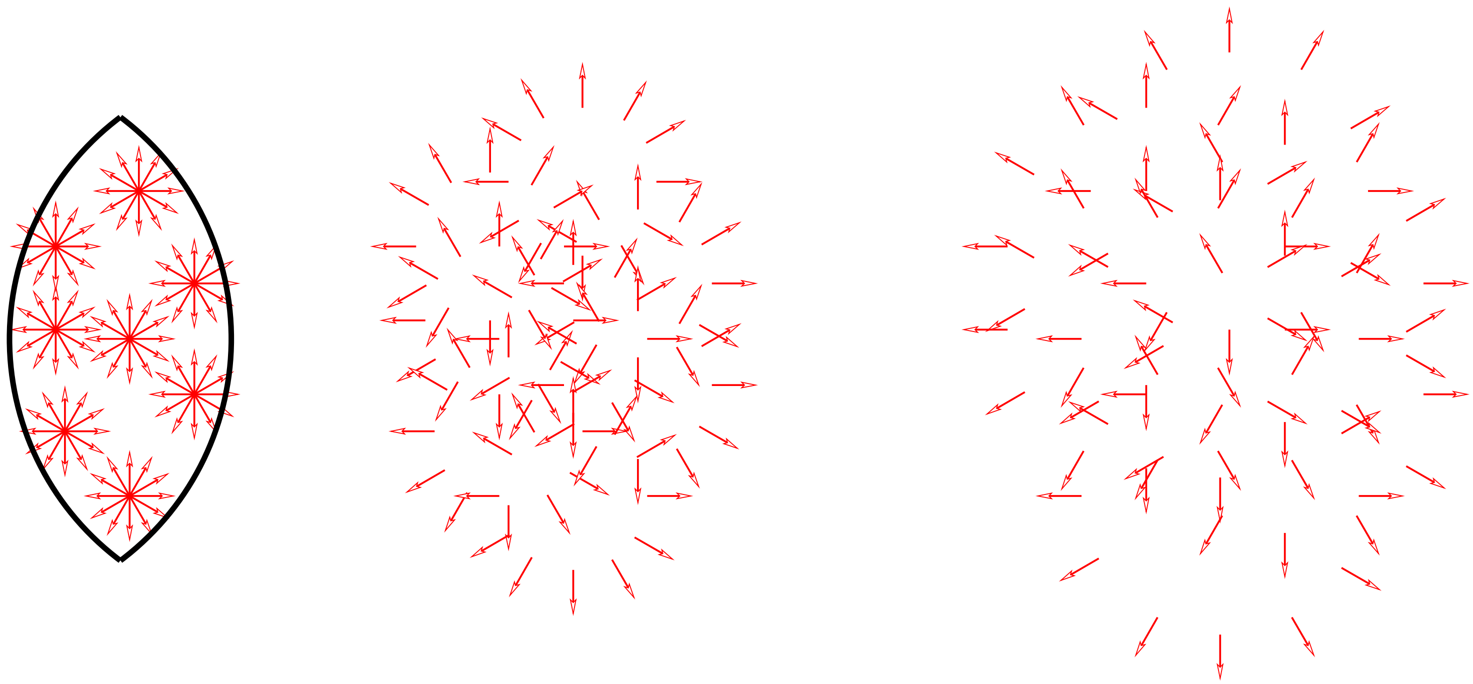
Nuclei generically strike off-center



leading to irregular shaped region of plasma

“Almond sliver” with long axis, short axis, and very short initial thickness along beam direction.

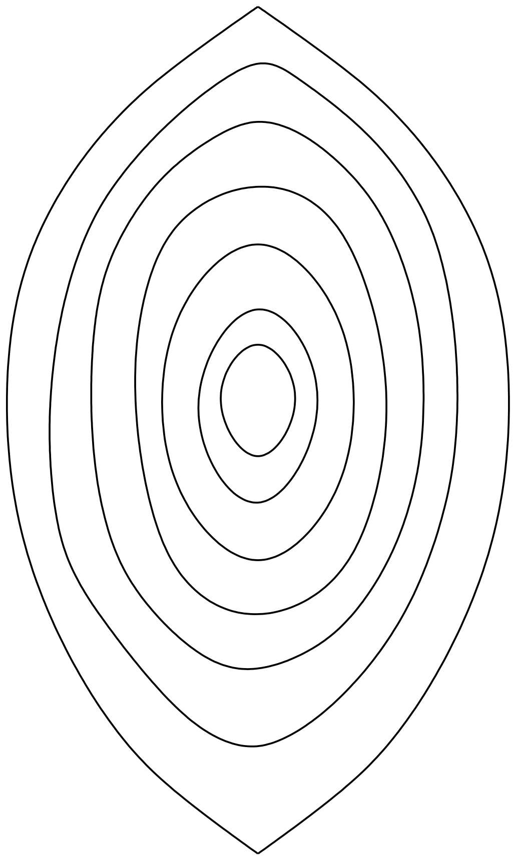
Behavior IF no re-interactions (transparency)

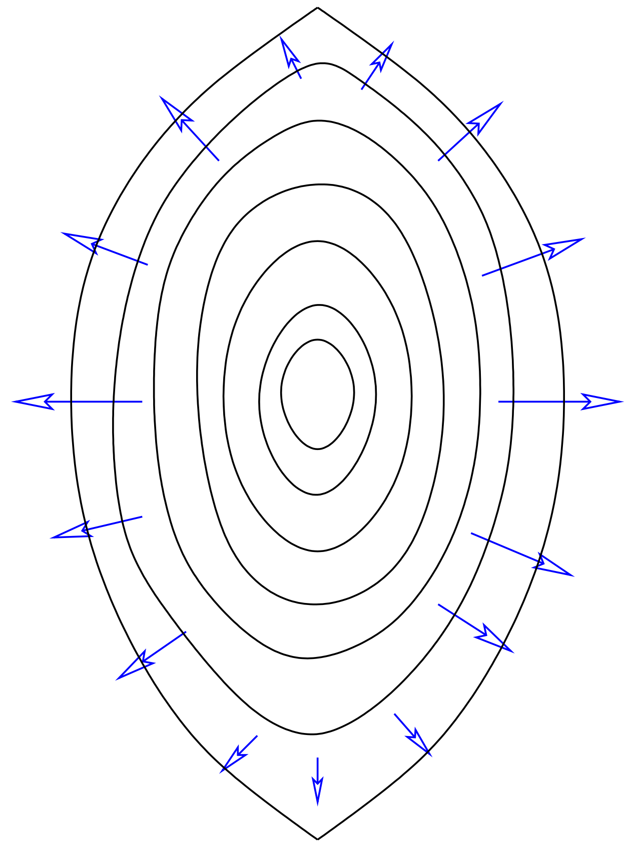


Just fly out and hit the detector.

Detector will see xy plane *isotropy*

local CM motions





Pressure contours   Expansion pattern

Anisotropy leads to anisotropic (local CM motion) flow.

Free particle propagation:

- System-average CM flow velocities $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Must have local CM $\langle p_x^2 \rangle < \langle p_y^2 \rangle$ so total $\langle p_x^2 \rangle = \langle p_y^2 \rangle$

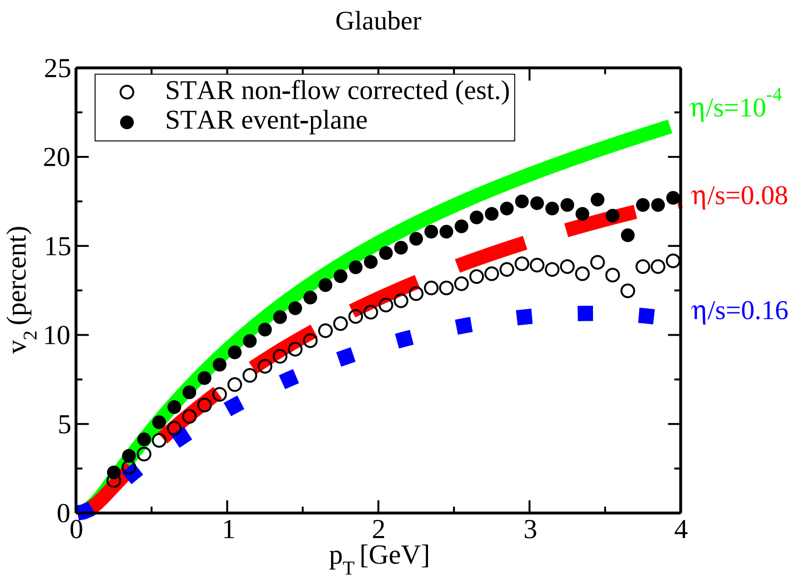
Efficient Equilibration:

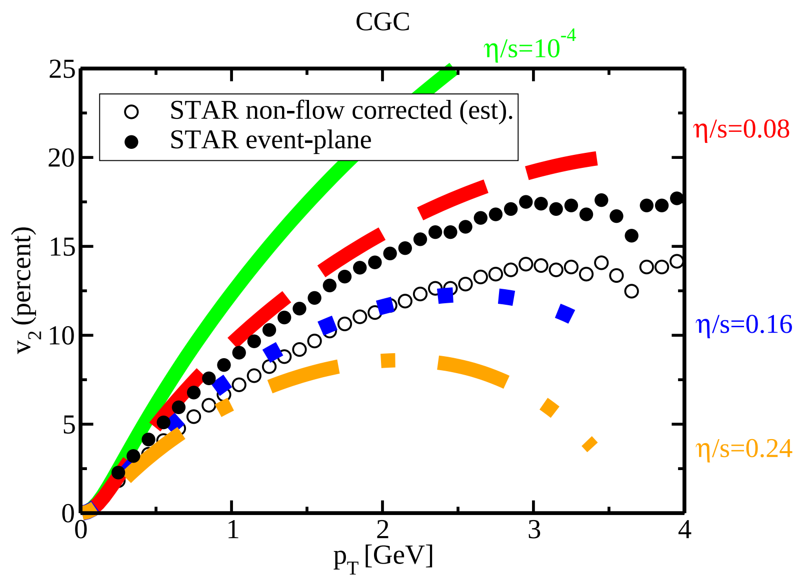
- System-average CM flow still has $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- system changes $locally$ towards $\langle T_{\mathrm{local\ CM}}^{xx} \rangle = \langle T_{\mathrm{local\ CM}}^{yy} \rangle$
- Adding these together, $\langle T_{\mathrm{tot,labframe}}^{xx} \rangle > \langle T_{\mathrm{tot,labframe}}^{yy} \rangle$

Net "Elliptic Flow" $v_2 \equiv \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2}$ measures re-interaction

Measured elliptic flow vs. theory fits

Hydrodynamic fits – based on assuming much rescattering





Luzum Romatschke 0804.4015, data STAR min-bias

Elliptic flow, differential in particle transverse momentum.

Two guesses at initial conditions (left and right),

Perfect rescattering (top) vs incomplete re-scattering (lower)

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \end{aligned}$$

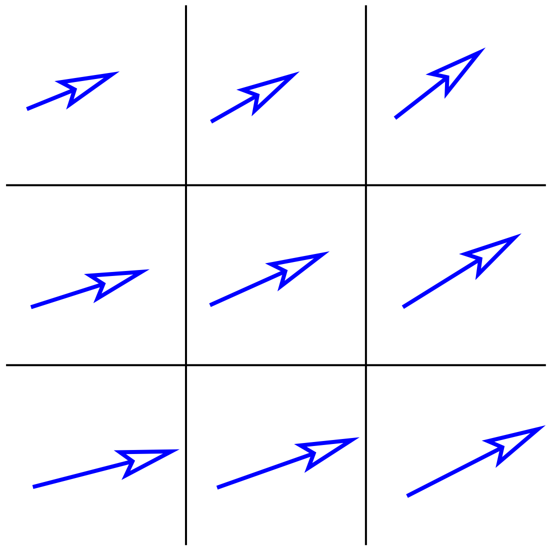
depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

works pretty well for heavy ions. But **quantify** corrections!

Nonideal Hydro

Each region feels information about neighboring regions diffusing across its boundary.

$\vec{v}$ nonuniformity means nonvanishing $\nabla_i v_j$ which will influence center region (diffusion of information)

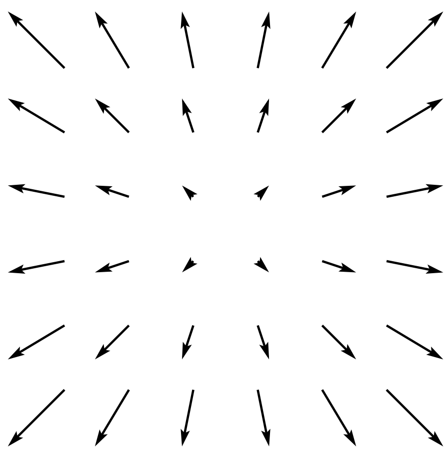


Decompose: scalar, antisymm, traceless symm tensor

$$\nabla_i v_j = \frac{\delta_{ij}}{3}\nabla\cdot v + \frac{1}{2}(\nabla_i v_j - \nabla_j v_i) + \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i - \frac{2\delta_{ij}}{3}\nabla\cdot v\right)$$

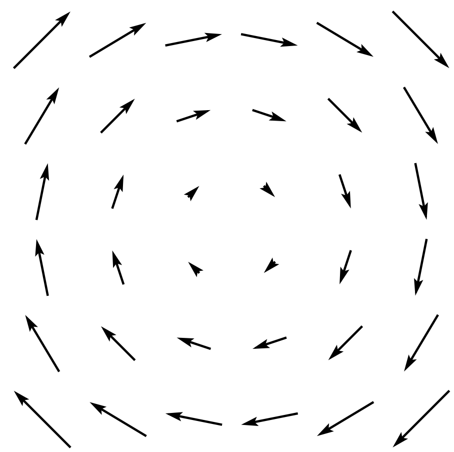
What each tensor piece means

$\delta_{ij}\nabla\cdot v$



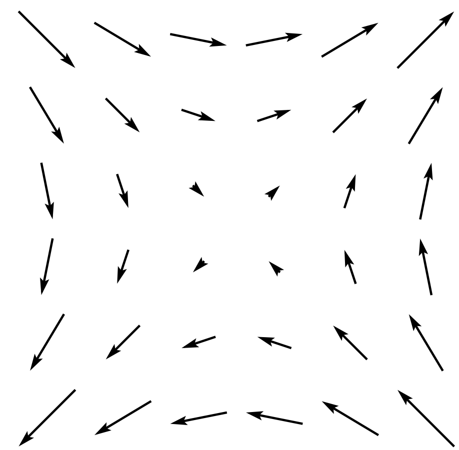
Divergence

$\nabla_i v_j - \nabla_j v_i$



Vorticity

$\nabla_i v_j + \nabla_j v_i - ..$



Shear flow

scalar divergence can change scalar pressure $P \Rightarrow P_{\text{equil.}} - \zeta\nabla\cdot v$

symm. tensor shear flow can change symm. tensor stress tensor

$T_{ij} \Rightarrow T_{ij,\text{equil.}} - \eta(\nabla_i v_j + \nabla_j v_i - ..)$

pseudovector vorticity cannot change either

Application to Nonequilibrium Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial,\epsilon,u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial u, \partial\epsilon) + \mathcal{O}(\partial^2 u, (\partial u)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \quad ?$$

Because in **relativisitc** setting, it is

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- Unstable: $v > c$ prop $+$ non-uniform flow velocity $\rightarrow$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem: short length scales, $\eta|\sigma| \sim P$. Numerics must treat these scales (or there's "numerical viscosity")

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi\, u^\alpha\partial_\alpha\sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \to -\Pi$ in order-2 term:

$$\tau_\pi\, u^\alpha\partial_\alpha\Pi^{\mu\nu} \equiv \tau_\pi\, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_π)

But why only one 2'nd order term???

Second order hydrodynamics

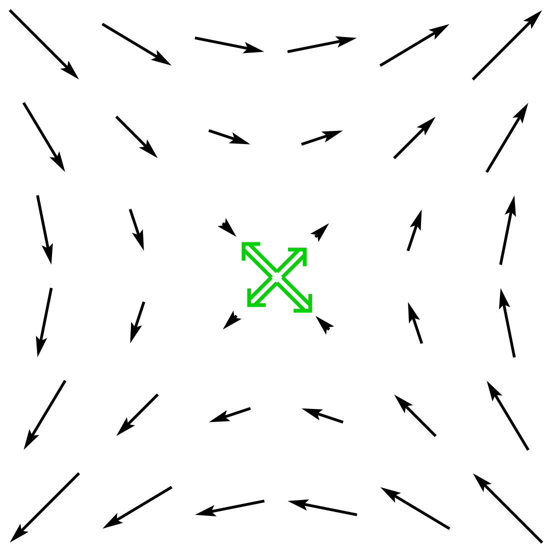
It is more consistent to include all possible 2'nd order terms.
Assume *conformality* and *vanishing chem. potentials*:
5 possible terms Baier *et al*, [arXiv:0712.2451]

$$\begin{aligned}\Pi^{\mu\nu}_{2\text{ ord.}} = \ & \eta\tau_\pi \left[u^\alpha \partial_\alpha \sigma^{\mu\nu} + \frac{1}{3}\sigma^{\mu\nu}\partial_\alpha u^\alpha\right] + \lambda_1 \left[\sigma^\mu_\alpha \sigma^{\nu\alpha} - (\text{trace})\right] \\ & + \lambda_2 \left[\frac{1}{2}(\sigma^\mu_\alpha \Omega^{\nu\alpha} + \sigma^\nu_\alpha \Omega^{\mu\alpha}) - (\text{trace})\right] \\ & + \lambda_3 \left[\Omega^\mu{}_\alpha \Omega^{\nu\alpha} - (\text{trace})\right] + \kappa \left(R^{\mu\nu} - \ldots\right) \ , \\ \Omega_{\mu\nu} \equiv \ & \frac{1}{2}\Delta_{\mu\alpha}\Delta_{\nu\beta}(\partial^\alpha u^\beta - \partial^\beta u^\alpha) \ \ [\text{vorticity}] \, .\end{aligned}$$

Let's learn what we can about this theory, its 6 coeff's

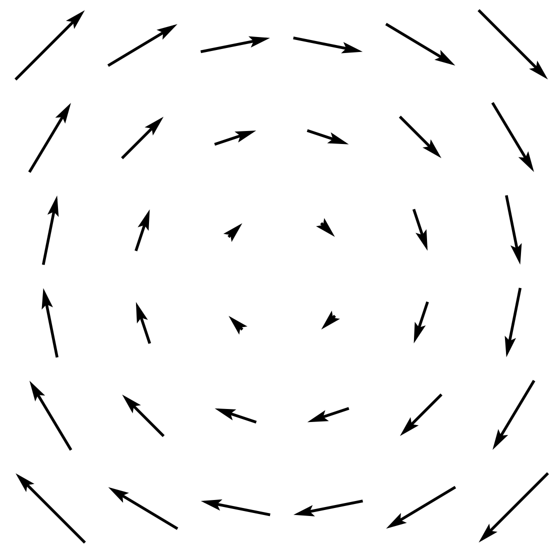
Step 1: What do $\sigma^{\mu\nu}$, $\Omega^{\mu\nu}$ mean?

Same-sign $\partial_x v_y = \partial_y v_x$



Shear flow

Opposite-sign $\partial_x v_y = -\partial_y v_x$

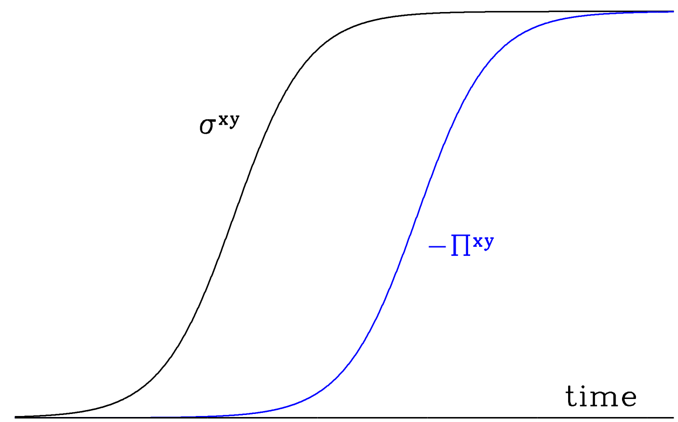


Vorticity

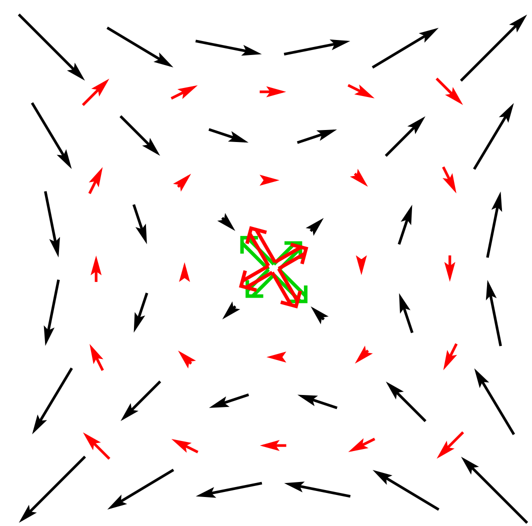
First order: Π_{xy} is symmetric. Can scale with $\sigma^{xy} = \partial_x v_y + \partial_y v_x$ but not with $2\Omega^{\mu\nu} = \partial_x v_y - \partial_y v_x$

Sign: fluid must "push back" against shear flow by stability!

τ_π: if shear flow $\sigma^{\mu\nu}$ "turns on", delay in $\Pi^{\mu\nu}$ "turning on"



λ_2: if shear makes $\Pi^{\mu\nu} \neq 0$, vorticity rotates $\Pi^{\mu\nu}$ axis from shear axis.
Sensible sign if $\lambda_2 < 0$ (sorry)



λ_1: some nonlinearity. λ_3: rotate about z axis$\rightarrow$ T^{zz} reduced

What do I expect these coefficients to be?

We can't calculate in full QCD at $T = 1.5T_c$:-(

We can calculate in two "toy" models:

- QCD in weak coupling

  [bravely extrapolate to realistic coupling]

- Analog theory, $\mathcal{N} = 4$ SYM at strong-coupling

  [bravely hope it is enough like QCD]

In $\mathcal{N} = 4$ SYM I find $\eta/s = 1/4\pi$.

In weak-coupling QCD I find ...

Perturbative QCD calculation

P. Arnold GM L. G. Yaffe 2003 compute with “souped-up” kinetic theory

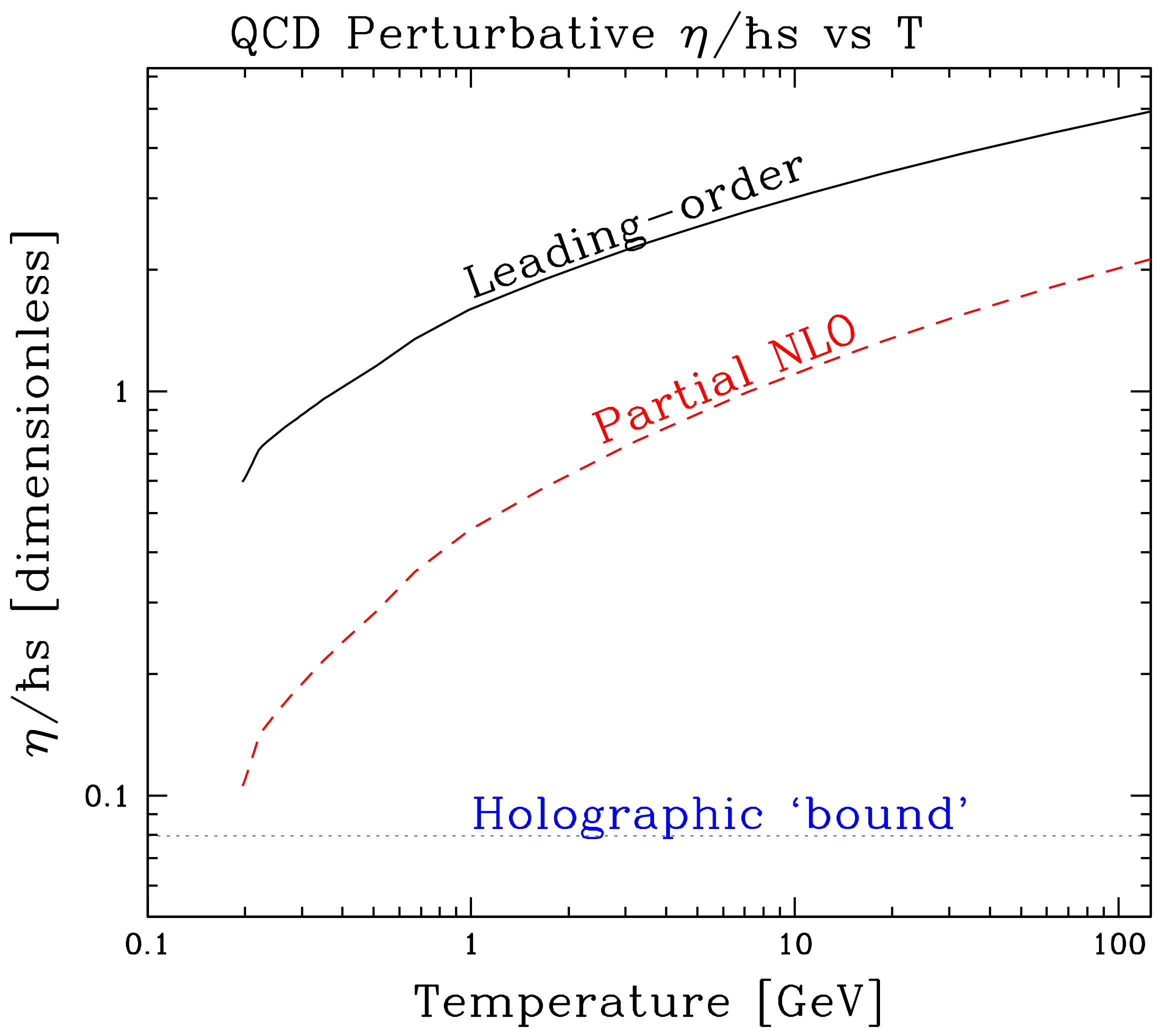
2 layers of effective field theory:

- Write down (effective) kinetic theory (Baym’s talk)
- One of matrix elements arises from (LPM) eff. thy: splitting due to partly-coherent soft scatterings

Needed just to get to leading-order in $\alpha_{\rm s}$

Range of validity / error estimates?
Requires NLO calculation. Now exists for some quantities, now know how to incorporate largest NLO ($\mathcal{O}(g_{\rm s})$) corrections (partial but not complete NLO calculation)

Perturbative results for η/s



Ouch! Pert thy seems to be very limited!!

2-order coefficients

Order-2 coefficients: similar poor behavior.

Certain *dimensionless ratios* are much more robust!

Ratio	QCD value	SYM value
$\frac{\eta\tau_\pi(\epsilon+P)}{\eta^2}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	4.1 to 5.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	0	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Arguably, we now know these ratios at the factor-of-2 level.

(Probably good enough for hydro!)

Kubo formulae

We want expressions which relate the transport coefficients to equilibrium correlation functions in the plasma fluct-diss

Would provide rigorous *definition* of $\eta, \lambda_{123}, \ldots$.

Example: long known that η is given by

$$\eta = \lim_{\omega \to 0} \frac{d}{d\omega} \int d^3x\, dt\, e^{i\omega t} \left\langle \Big[T^{xy}(x,t)\, ,\, T^{xy}(0,0) \Big] \right\rangle \Theta(t)$$

Similar relations for second-order transport coefficients?

How to get Kubo relations

Find framework where I can compute $T^{\mu\nu}$ using hydro or using field theory, both should be valid.

Time-varying geometry does the job:

- Start at $t \ll 0$ with flat-space, equilibrium thermal system $\rho = e^{-HT}$, $g_{\mu\nu} = \eta_{\mu\nu}$
- At some time $t_0 < 0$ start deforming metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ in such a way as to force the system to experience shear and vorticity
- Choose $h_{\mu\nu}$ small and slowly varying so you stay near equilibrium and gradient expansion, hydro are valid

Give a hydro theorist $h_{xy}(z,t)$, $h_{0x}(y)$ nonzero.

Ask them what $T^{\mu\nu}(0)$ will be.

Answer: $$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu}$$

First, find ϵ, u: Hydro says

$$\nabla_\mu T^{\mu\nu} = 0 \quad \to \quad u^\mu = (1,0,0,0) + \mathcal{O}(\partial^2)\,.$$

Then $u_\mu = (1, h_{0x}, 0, 0)$, $\Gamma^x{}_{yt}$ *etc* nonzero.
They give rise to nonzero σ^{xy}, Ω^{xy}, etc:

$$\sigma^{xy} = \partial_t h_{xy}\,, \qquad \Omega^{xy} = -\partial_y h_{0x}/2$$

Other terms $R^{\langle xy\rangle}$, $u\cdot\nabla\sigma^{xy}$ found similarly.

T^{xy} at $\mathcal{O}(h)$ and $\mathcal{O}(\partial^2)$, for $h_{xy} \neq 0$:

$$T^{xy} = -\eta\partial_t h_{xy} + \eta\tau_\pi\partial_t^2 h_{xy} - \frac{\kappa}{2}\left(\partial_t^2 h_{xy} + \partial_z^2 h_{xy}\right)$$

and T^{xy} at $\mathcal{O}(\partial^2, h^2)$ for $h_{xz}(t)$, $h_{yz}(t)$, $h_{x0}(z)$, $h_{y0}(z)$ nonzero:

$$\begin{aligned}\Pi^{xy} = {} & \eta\partial_t(h_{xz}h_{yz}) + \frac{\kappa}{2}\left(h_{xz}\partial_t^2 h_{yz} + h_{yz}\partial_t^2 h_{xz}\right) + \lambda_1\partial_t h_{xz}\partial_t h_{yz} \\ & +\eta\tau_\pi\left(\frac{1}{2}\partial_t h_{xz}\partial_z h_{0y} + \frac{1}{2}\partial_t h_{yz}\partial_z h_{0x}\right. \\ & \left. - \partial_t h_{xz}\partial_t h_{yz} - h_{xz}\partial_t^2 h_{yz} - h_{yz}\partial_t^2 h_{xz}\right) \\ & -\frac{\lambda_2}{4}\left(\partial_t h_{xz}\partial_z h_{0y} + \partial_t h_{yz}\partial_z h_{0x}\right) + \frac{\lambda_3}{4}\partial_z h_{0x}\partial_z h_{0y}\end{aligned}$$

So at $\mathcal{O}(h)$ T^{xy} depends on η, τ_π, κ; at $\mathcal{O}(h^2)$, depends on all 6!

Give field theorist $h_{xy}(z,t)$, etc nonzero.

Ask them what T^{xy} will be.

$$\langle T^{\mu\nu}(t)\rangle = \mathrm{Tr}\, e^{-HT} e^{iHt} \hat{T}^{\mu\nu} e^{-iHt}, \quad T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathcal{L}}{\partial h_{\mu\nu}}$$

with $H = H[h(t')]$! Schwinger-Keldysh in $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$W \equiv \ln \int_{C=} \mathcal{D}(\Phi_1, \Phi_2, \Phi_3)\; e^{iS_1[h_1,\Phi_1] - iS_2[h_2,\Phi_2] - S_3[\Phi_3]}$$

$S_1[h_1]$, $S_2[h_2]$ depend on independent fields *and metrics!*

$$T_1 = \frac{-2i\delta W}{\delta h_1}, \qquad T_2 = \frac{+2i\delta W}{\delta h_2}$$

Introduce average and difference variables:

$$h_r = \tfrac{h_1+h_2}{2}\,,\ h_a = h_1 - h_2\,, \quad T_r = \tfrac{T_1+T_2}{2}\,,\ T_a = T_1 - T_2$$

Note, due to signs $e^{iS_1 - iS_2}$, $T_r = \frac{-2i\delta W}{\delta h_a}\,, \quad T_a = \frac{-2i\delta W}{\delta h_r}$. Take $\delta/\delta h_a \to \langle T\rangle$. Then set $h_a = 0$, $h_r = h$, expand in h:

$$\begin{aligned} \langle T^{\mu\nu}\rangle_h &= G_r^{\mu\nu}(0) - \frac{1}{2}\int d^4x G_{ra}^{\mu\nu,\alpha\beta}(0,x) h_{\alpha\beta}(x) \\ &\quad + \frac{1}{8}\int d^4x d^4y G_{raa}^{\mu\nu,\alpha\beta,\gamma\delta}(0,x,y) h_{\alpha\beta}(x) h_{\gamma\delta}(y) \end{aligned}$$

$$\begin{aligned} G_{ra\ldots}^{\mu\nu,\alpha\beta\ldots}(0,x\ldots) &\equiv \left.\frac{(-i)^{n-1}(-2i)^n \delta^n W}{\delta g_{a,\mu\nu}(0)\delta g_{r,\alpha\beta}(x)\ldots}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \\ &= (-i)^{n-1}\left\langle T_r^{\mu\nu}(0) T_a^{\alpha\beta}(x)\ldots\right\rangle + \text{c.t.} \end{aligned}$$

Equate: $T^{\mu\nu}_{\rm hydro} = T^{\mu\nu}_{\rm field\ theory}$ (Matching calculation!)
Use that h slowly varying, find BRSSS 0712.2451

$$\begin{aligned}
\eta &= -i\partial_\omega G^{xy,xy}_{ra}(\omega,k)\big|_{\omega=0=k}\,, \\
\kappa &= -\partial^2_{k_z} G^{xy,xy}_{ra}(\omega,k)\big|_{\omega=0=k}\,, \\
\eta\tau_\pi &= \frac{1}{2}\Big(\partial^2_\omega G^{xy,xy}_{ra}(\omega,k) - \partial^2_{k_z} G^{xy,xy}_{ra}(\omega,k)\Big)\Big|_{\omega=0=k}\,.
\end{aligned}$$

And at nonlinear order,

$$\begin{aligned}
\lambda_1 &= \eta\tau_\pi - \lim_{p^t,q^t\to 0} \frac{\partial^2}{\partial p^t \partial q^t} \lim_{\mathbf{p},\mathbf{q}\to 0} G^{xy,xz,yz}_{raa}(p,q) \\
\lambda_2 &= 2\eta\tau_\pi - 4 \lim_{p^t,\mathbf{q}\to 0} \frac{\partial^2}{\partial p^t \partial q^z} \lim_{\mathbf{p},q^t\to 0} G^{xy,xz,0y}_{raa}(p,q) \\
\lambda_3 &= -4 \lim_{\mathbf{p},\mathbf{q}\to 0} \frac{\partial^2}{\partial p^z \partial q^z} \lim_{p^t,q^t\to 0} G^{xy,0x,0y}_{raa}(p,q)\,.
\end{aligned}$$

Nature of κ and λ_3

κ and λ_3 have Kubo relations $\mathbf{NOT}$ involving ∂_t's.

May (must!) set frequency $\omega = 0$ from outset:

$$\begin{aligned}
\kappa &= -\lim_{\vec{q}\to 0} \frac{\partial^2}{\partial q_z^2} G_{ra}^{xy,xy}(\vec{q},\omega=0) \\
\lambda_3 &= -6 \lim_{\vec{p},\vec{q}\to 0} \frac{\partial^2}{\partial p_y \partial q_y} G_{raa}^{xx,0x,0x}(\vec{p},\omega_p=0,\vec{q},\omega_q=0)
\end{aligned}$$

But $G_{ra\ldots}(\omega=0) = (-)^{n-1} G_{\mathrm{E}}(\omega_{\mathrm{E}}=0)$ Euclidean func.

Weak-coupling expansions: $\kappa, \lambda_3 = T^2(\mathcal{O}(1) + \mathcal{O}(g, g^2, \ldots))$

Leading weak-coupling values calculable and *nonzero*

But is hydro even consistent?

We said $\Pi^{\mu\nu} = \mathcal{O}(\partial u) + \mathcal{O}(\partial^2 u, (\partial u)^2) + \ldots$

based on assumption thermalization is local, microscopic.

Hydro itself predicts long-lived shear,sound modes:

$$0 = \partial_\mu \Big(T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \eta \sigma^{\mu\nu} \Big)$$

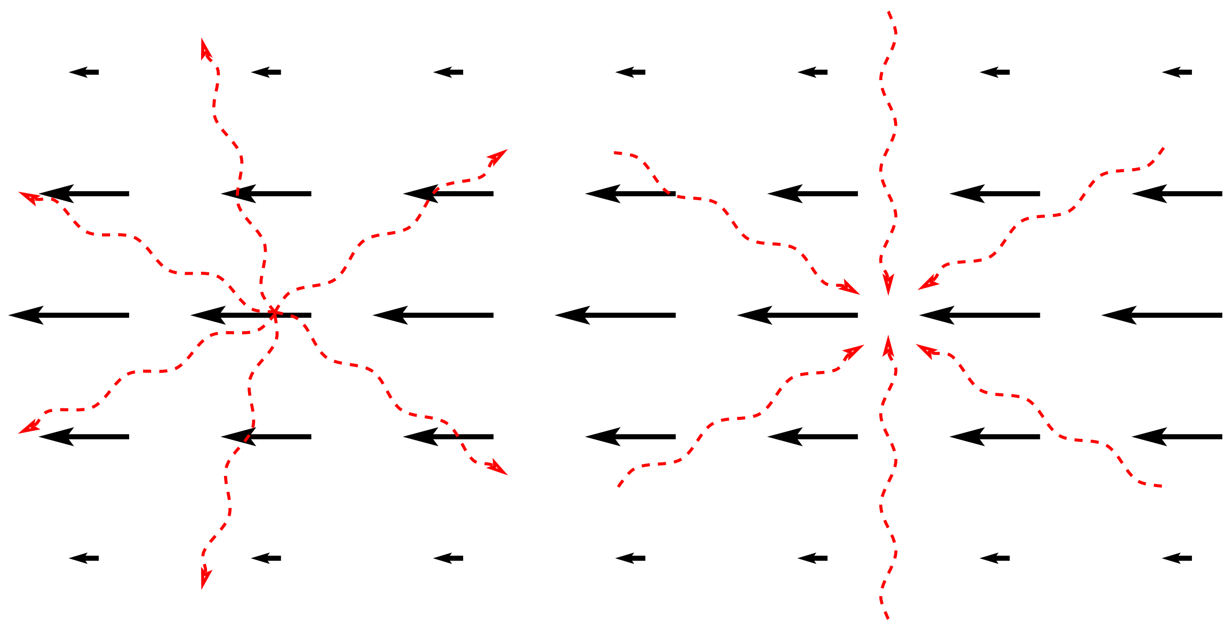
fluctuations in u^μ, ϵ: dispersion relations

$$\omega_{\rm shear} = i \frac{\eta}{\epsilon + P} k^2 \,, \qquad \omega_{\rm sound} = \pm \frac{k}{\sqrt{3}} + i \frac{2\eta}{3(\epsilon + P)} k^2$$

Small k: long lived, dissipation *not* local,microscopic

Hydro Waves Contribute to Viscosity!

Consider shear flow:



Shear: transport of x-momentum from middle to edge.

One mechanism: propagation of hydro (sound) waves!

η etc are Wilson coeffs. Do the RG flow!

How to compute hydro contribution to hydro

Above we found

$$-i\langle T_r^{xy} T_a^{xy} \rangle = G_{ra}^{xy,xy}(\omega) = P - i\eta\omega + \eta\tau_\pi\omega^2 + \ldots$$

Calculate contrib. of hydro modes themselves to G^{xyxy}.

$$\begin{aligned} \text{Feynman rules:} \quad T^{ij} &= (\epsilon + P)u^i u^j + P g^{ij}\,, \\ \langle u^i u^j(k,\omega)\rangle &= \frac{T}{\epsilon+P}\,\frac{(\delta^{ij} - \hat{k}^i\hat{k}^j)2\gamma_\eta k^2}{(\gamma_\eta k^2 - i\omega)(\gamma_\eta k^2 + i\omega)}\,\text{shearwave} \\ \left[\gamma_\eta = \tfrac{\eta}{\epsilon+P}, \gamma'_\eta = \tfrac{4}{3}\gamma_\eta\right] \qquad & + \frac{T}{\epsilon+P}\,\frac{(\hat{k}^i\hat{k}^j)2\gamma'_\eta k^2\omega^2}{(\omega^2 - k^2/3)^2 + (\gamma'_\eta k^2\omega)^2}\,\text{soundwave} \end{aligned}$$

Compute $T_{xy}T_{xy}$ using these expressions.

Computing $G_{ra}^{xy,xy}(\omega, k=0)$

Contribution of hydro modes up to cutoff $k_{\rm max}$

$$G_{ra}^{xy,xy}(\omega)[\text{hydro}] = -i\omega\left(\frac{17Tk_{\rm max}}{120\pi^2\gamma_\eta}\right) + (i+1)\omega^{\frac{3}{2}}\frac{7+\left(\frac{3}{2}\right)^{\frac{3}{2}}T}{240\pi\gamma_\eta^{3/2}}$$

$k_{\rm max}$: k-scale above which hydro incorrect/inconsistent.
Small η/s: larger $k_{\rm max}$ *and* larger contrib.s (hydro waves live longer)

- $-i\omega$ term: extra contrib. to η
- $i\omega^{3/2}$: effective ω dependence of η.
- $\omega^{3/2}$: like τ_π but *wrong* ω dependence.

Lesson: η

Small η: freer propagation of sound, shear modes.

More momentum transport by hydro waves, raising η.

Depends on $k_{\rm max}$. Where does hydro break down?

Scale where it's no longer self-consistent.

Safe guess: $k_{\rm max} < \tau_\pi^{-1}/2$. In $\mathcal{N}{=}4$ SYM, this is about $2T$.

- $\mathcal{N}{=}4$SYM: added η/s is $\sim 1/N_{\rm c}^2$.
- Weak coupling: $\eta_{\rm from\ hydro} \sim \alpha^4$ while $\eta_{\rm tot} \sim \alpha^{-2}$
- Real QCD: $\frac{\eta}{s} = .16$: add 0.01. $\frac{\eta}{s} = .08$: add 0.036!

Lesson: τ_π

Weak coupling and large N_c: comparing

$$N_\mathrm{c}^0 \alpha^3 T^{5/2}\, \omega^{3/2} \quad \text{vs} \quad N_\mathrm{c}^2 \alpha^{-4} T^2\, \omega^2$$

Deep IR, $\omega^{3/2}$ term wins, 2-order hydro breaks.
But scale where $\omega^{3/2}$ term takes over is $\omega \sim N_\mathrm{c}^{-4} \alpha^{14} T$.

Check that ω where they equal is more IR than "your physics" and then use 2-order hydro!

- $N_\mathrm{c} = 3 = N_\mathrm{f}$ QCD, $T = 200\mathrm{MeV}$, $\frac{\eta}{s} = .16$: $\omega \sim \frac{T}{20}$ Safe!
- $N_\mathrm{c} = 3 = N_\mathrm{f}$ QCD, $T = 200\mathrm{MeV}$, $\frac{\eta}{s} = .08$: $\omega \sim 7T$ Problem!

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Need 2'nd order Hydro, 6 hydro coefficients!
- Weak coupling methods fail below $T = 100$GeV.
  But some dimensionless ratios are robust.
- Kubo relations for nonlinear coefficients found.
  κ, λ_3 special (really thermodynamic)
- Hydro waves contribute to hydro coefficients!
- Self-consistency issues if η too small, and very low freq.