Self-Similarity And The Random Walk

Lecture: Phase Transitions And Renormalization Group

Presenter: Theresa Lindner

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Agenda

- Motivation
- Definitions
- Cantor sets
    - One or more scaling factors
- Fractals in higher dimensions
- The Random Walk
    - Fractal dimension
    - Gaussian distribution
    - Drift term

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Motivation

- Scale invariance
  - Magnification by suitable scale factor
  - Identical object
  - Self-similarity on all scales

- Critical behaviour
  - Loss of scale
  - Divergence of correlation length
  - Perturbation expansion fails

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Definitions

- Euclidian dimension d
  - Dimension of observed space

- Topological dimension d_T
  - Dimension of observed object

- Fractal (Hausdorff-) dimension
  - Cover object in d-dimensional balls of radius a
  - $a \to 0$
  - $N(a) \sim a^{-D}$

$$\boxed{d_T \leq D \leq d}$$

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Cantor Sets - One Scaling Factor

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- Divide each line segment a_0 into three of $\frac{a_0}{3}$.
- Cover object in line segments of length a.

$$\boxed{d = 1, d_T = 0}$$

- Approach 1: $a = a_0 3^{-n}$ in step n $\leftrightarrow n = \frac{\ln(a/a_0)}{\ln 3}$

$$N(a) = 2^n = \left(\frac{a}{a_0}\right)^{-ln2/ln3} \sim a^{-D}$$

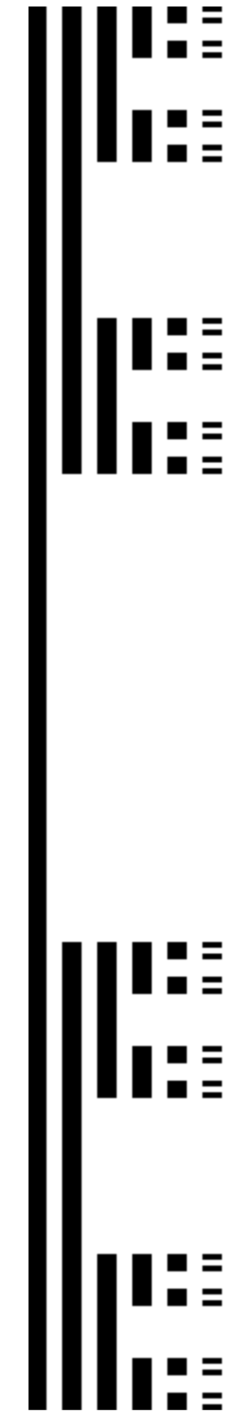
$$\boxed{D = \frac{ln2}{ln3} \approx 0.6309}$$

- Approach 2:

$$N(a) = 2N(3a) = 4N(9a) = \ldots$$

$$a^{-D} = 2 \cdot 3^{-D} \cdot a^{-D}$$

$$\boxed{D = \frac{ln2}{ln3} \approx 0.6309}$$



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Cantor Sets - Two Scaling Factors

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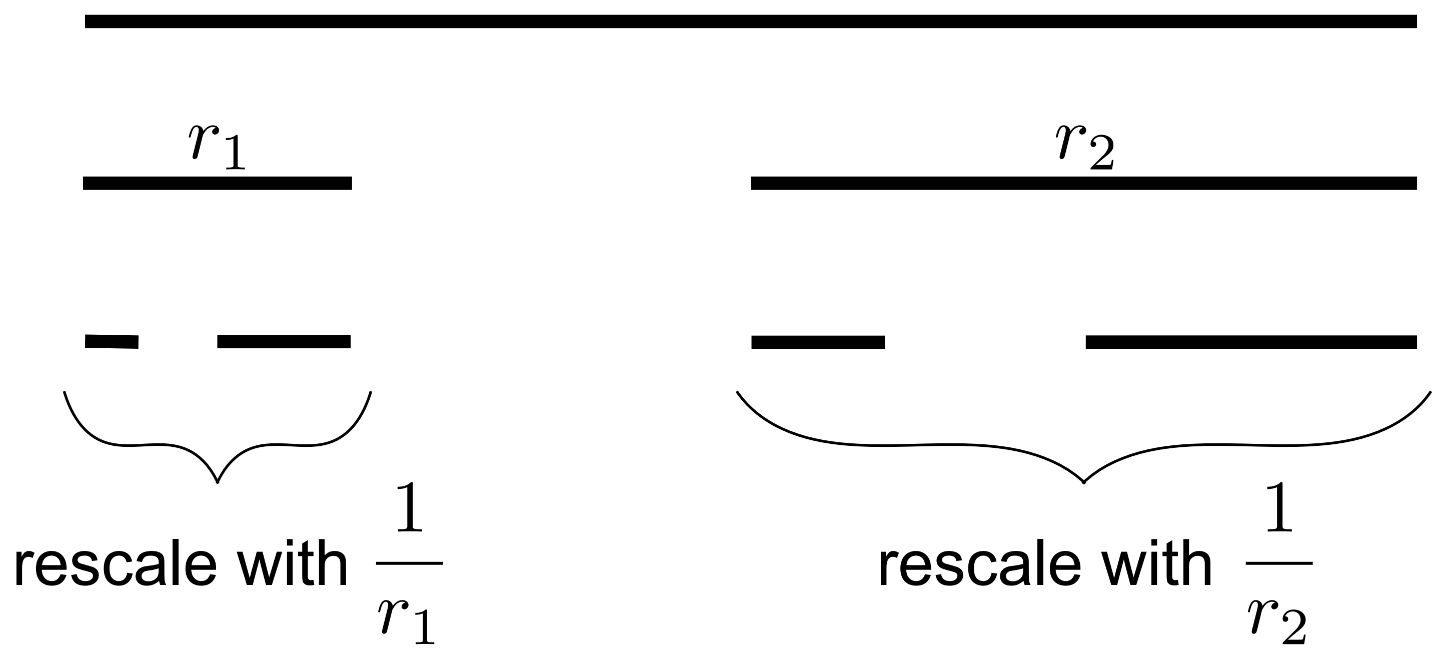
- Generator with more than one element and different scaling factors r_i ($\sum r_i < 1$)



$$N(a) = N(\frac{a}{r_1}) + N(\frac{a}{r_2}) \rightarrow \boxed{1 = \sum_{j=1}^{N} r_j^D}$$

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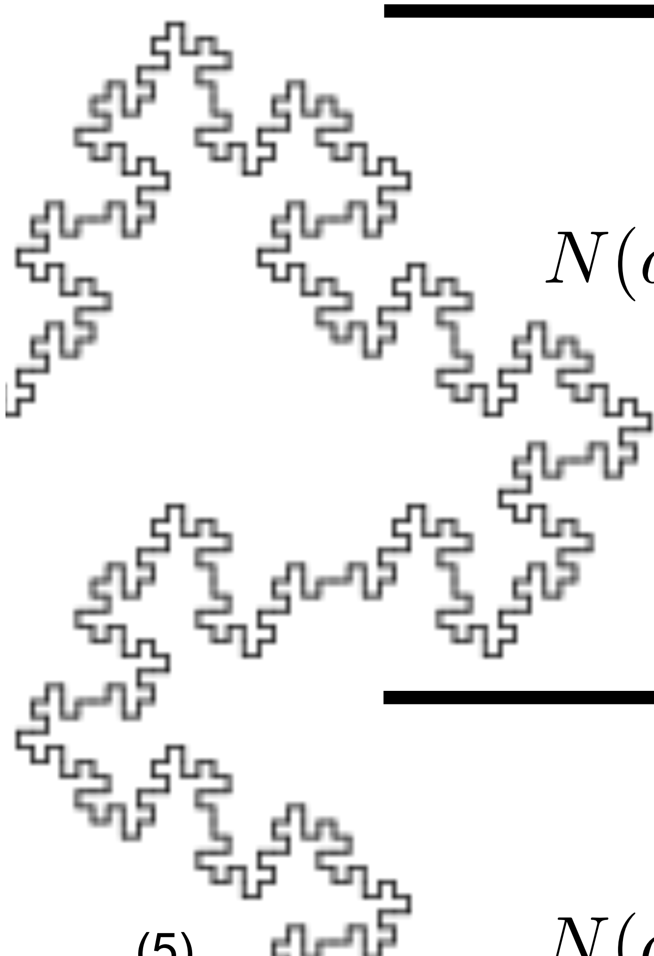
Koch Islands - Fractals In Higher Dimensions



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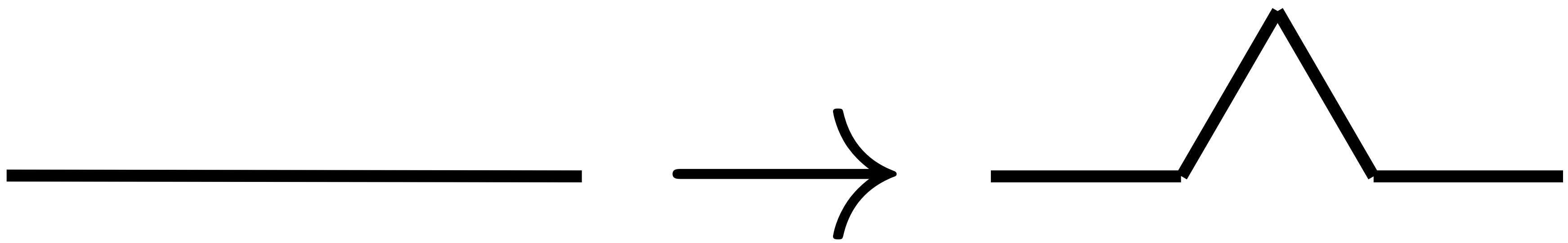
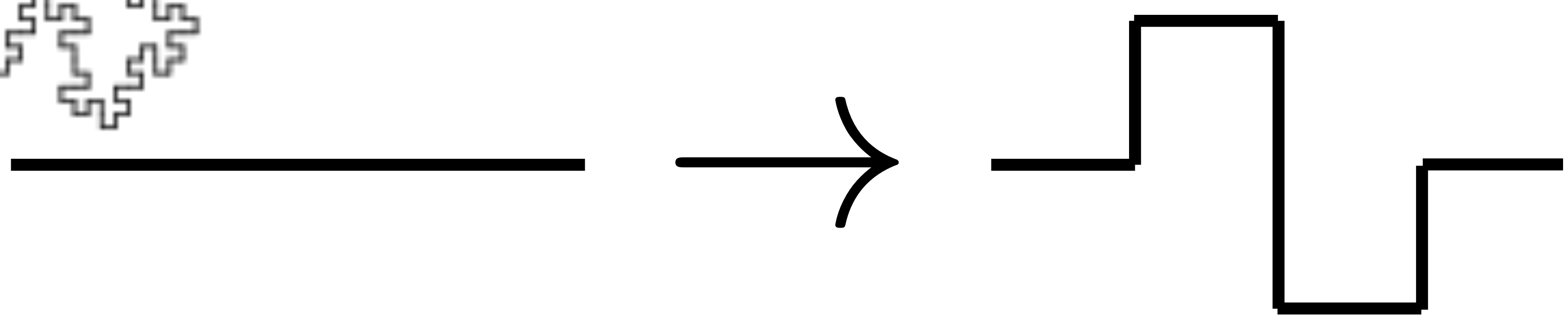


$$N(a) = 4N(3a) \qquad D = \frac{ln4}{ln3} \approx 1.262$$



$$N(a) = 8N(4a) \qquad D = \frac{ln8}{ln4} = \frac{3}{2}$$

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The Random Walk

- Brownian motion
- Diffusion
- Conformation of linear chain molecules

- End point: $\vec{R} = \sum_{i=1}^{M} \vec{r_i}$

$$\langle \vec{R} \rangle = 0 \qquad \langle |\vec{R}|^2 \rangle = M a_0^2 \qquad \boxed{R = \sqrt{M} a_0}$$

The Random Walk - Fractal Dimension

- Divide RW in $\frac{M}{n}$ subwalks of n steps
- $r(n) = \sqrt{n}a_0$ for each subwalk
- Coarse grained RW with $\frac{M}{n}$ steps of length $a = \sqrt{n}a_0$
- Assumption: $N(a_0) \sim M \quad \rightarrow \quad N(\sqrt{n}a_0) \sim \frac{M}{n} \sim \frac{1}{n}N(a_0)$

$$N(\sqrt{n}a_0) \sim \frac{M}{n} \sim \frac{1}{n}N(a_0) \leftrightarrow nN(\sqrt{n}a) = N(a_0)$$

$$\boxed{N(a) \sim a^{-2} \quad \text{for} \quad d \geq 2}$$

The Random Walk - Gaussian Distribution

- a_0 constant $\rightarrow$ Gaussian distribution

$$p(\vec{r}) = (2\pi\sigma_0^2)^{-d/2} \exp\left[-\frac{|\vec{r}|^2}{2\sigma_0^2}\right]$$

- Coarse graining
- Rescaling

$$\vec{r} \rightarrow \vec{r}\,' \rightarrow \vec{r}\,''$$

The Random Walk - Gaussian Distribution

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- Coarse graining $\vec{r} \to \vec{r}\,'$

$$\vec{r}\,' = \sum_{i=1}^{n} \vec{r}_i$$

$$P(\vec{r}\,') = \int d\vec{r}_1 ... d\vec{r}_n \delta\left(\vec{r}\,' - \sum_{i=1}^{n} \vec{r}_i\right) p(\vec{r}_1)...p(\vec{r}_n)$$

$$= \int \frac{d^d k}{(2\pi)^d} \exp\left[-n|\vec{k}|^2 \frac{\sigma_0^2}{2} - i\vec{k}\cdot\vec{r}\,'\right]$$

$$= (2\pi n \sigma_0^2)^{-d/2} \exp\left[-\frac{|\vec{r}|^2}{2n\sigma_0^2}\right]$$

The Random Walk - Gaussian Distribution

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- Rescaling $\vec{r}\,' \to \vec{r}\,''$

$$\vec{r}\,' = \sqrt{n}\vec{r}\,''$$

$$P(\vec{r}\,')d^d r' = P(\vec{r}\,'')d^d r''$$

$$P(\vec{r}\,')n^{d/2}d^d r'' = P(\vec{r}\,'')d^d r''$$

$$P(\vec{r}\,'') = (2\pi\sigma_0^2)^{-d/2}\exp\left[-\frac{|\vec{r}\,''|^2}{2\sigma_0^2}\right]$$

$$\boxed{\sigma^{(R)} = \sigma_0}$$

The Random Walk - Gaussian Distribution

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- Consider RMS distance $R(M)$
- Dimensional analysis: $R(M,\sigma) \sim \sigma M^{\nu}$
- Coarse graining: $\sigma M^{\nu} = \sqrt{n}\sigma(\frac{M}{n})^{\nu}$ $\boxed{\nu = \frac{1}{2}}$

$$\boxed{\text{Scaling exponent} \quad \nu = \frac{1}{D}}$$

The Random Walk - Adding A Drift

- Add a drift term $\vec{r}_0$

$$p(\vec{r}) = (2\pi\sigma^2)^{-d/2} \exp\left[-\frac{|\vec{r}-\vec{r}_0|^2}{2\sigma^2}\right]$$

- Coarse graining: $\sigma' = \sqrt{n}\sigma$

$${\vec{r}_0}' = n\vec{r}_0$$

- Rescaling: $\sigma^{(R)} = \sigma$

$${\vec{r}_0}^{(R)} = \sqrt{n}\vec{r}_0$$

- Fixed points: $(\sigma, \vec{r}_0 = 0)$ and $(\sigma, \vec{r}_0 \to \infty)$

The Random Walk

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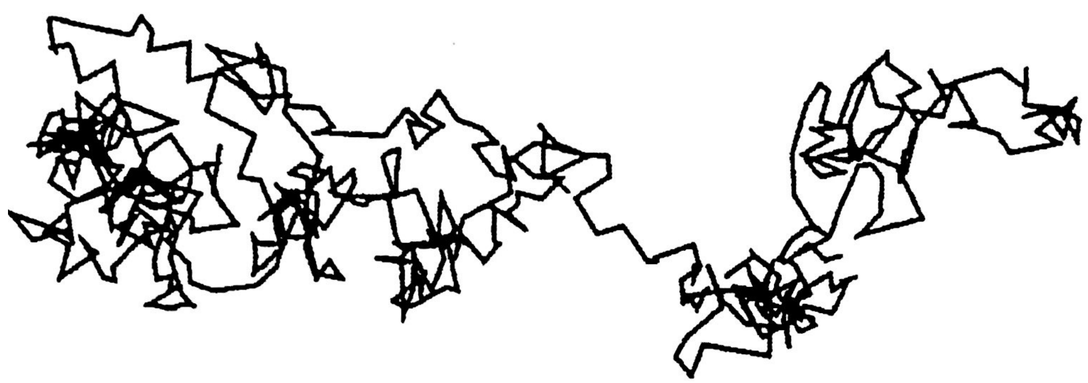
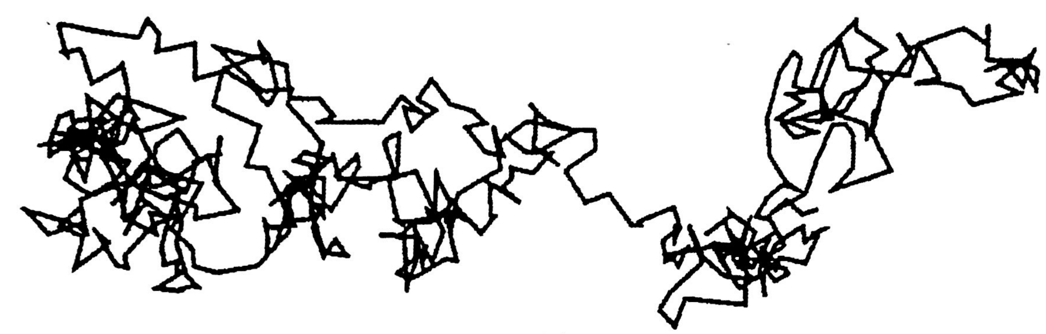


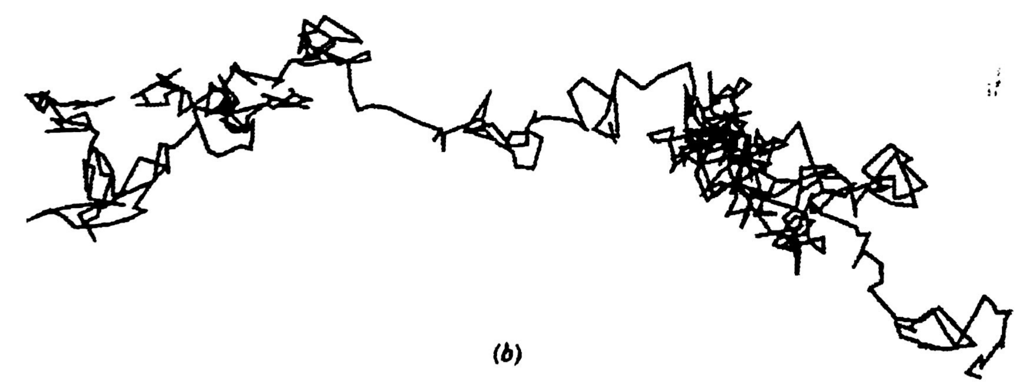
(a)

(b)

Without drift: Self-similarity



(a)



(b)

(c)

With drift: Appearence of 1D-walk

Sources

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