Kai Hebeler
Institut für Kernphysik

TECHNISCHE UNIVERSITÄT DARMSTADT

Phase transitions and the Renormalization Group

Summer term 2017
Problem set 1
Discussion of problems: Monday, May 8

April 26, 2017

Problem H1: Phase space and partition functions

In this problem we study phase space integrals and investigate the equivalence of the microcanonical and canonical ensemble for a free classical gas.

1. Show that the surface S_N of an N-dimensional sphere with a radius R is given by

$$S_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} R^{N-1} \tag{1}$$

   Hint: Use the integral

$$\int d^N a\, e^{-a_1^2-\ldots-a_N^2} = \left(\int_{-\infty}^{\infty} da_1 e^{-a_1^2}\right)^N = \pi^{N/2}$$

   and the integral representation of the Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

2. Calculate the volume of a spherical shell of thickness ΔR. Determine the fraction of the volume of this shell for $\Delta R/R = 0.01$ compared to the volume of the full sphere for N=3, 10, 100, 500 and 1000 dimensions.

3. Evaluate the semiclassical partition function in the microcanoinical ensemble for an ideal gas in a volume V, with total energy E and N particles:

$$Z_{mc} = \int \frac{d^{3N}x\, d^{3N}p}{h^{3N}N!} \left[\Theta(E - H(p,x)) - \Theta(H(p,x) - (E - \Delta E))\right] \tag{2}$$

   and derive the equation of state for the ideal gas:

$$PV = Nk_BT. \tag{3}$$

   How do you choose the value of ΔE?

HINT: Use the Stirling relation for large N (can you derive this relation?):

$$N! \sim N^N e^{-N}$$

4. Calculate now the canonical partition function in semiclassical approximation for an ideal gas and show that you obtain the same thermodynamic results like in the microcanonical ensemble.

Problem H2: Legendre transformation

The Legendre transformation allows to perform a variable transformation from one set of variables to a set of *canonical* variables. Consider as an example a function of two variables $f(x, y)$.

1. Consider the total derivative df. For this case there are two pairs of conjugated variables:

$$x \Leftrightarrow a \equiv \left.\frac{\partial f}{\partial x}\right|_y , \quad y \Leftrightarrow b \equiv \left.\frac{\partial f}{\partial y}\right|_x . \tag{4}$$

   Show that the function $g = f - ax$ is a natural function of the variables y and a, whereas $h = f - by$ is a function of x and b.

2. Show that in classical mechanics the Hamilton function $H(q, p)$ is the Legendre transform of the Lagrange function $L(q, \dot{q})$ and the free energy $F(T, V)$ the Legendre transform of the energy E in statistical physics.

3. Consider now a function depending on N variables. How many Legendre-transforms can be in principle constructed? In which cases is the Legendre-transformed function a single-values function?

4. Consider as an example the function $f(x) = e^{x-1}$. Calculate explicitly the Legendre transform. Illustrate how to determine the Legendre-transformed function graphically. What is the interval of definition of the function?