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Phase transitions and the Renormalization Group

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Problem set 2

Discussion of problems: Wednesday, May 24

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Problem 3: Van der Waals equation and the law of corresponding states

In the lecture we derived the Van der Waals equation of state by taking into account an excluded volume of size b per particle due to the hard-core repulsion of particles at short distances and a long-range mean field interaction term charaterized by the parameter a:

$$F(V,T) = -Nk_BT\log(V-Nb) - a\frac{N^2}{V}, \quad \text{i.e.} \quad P = \frac{Nk_BT}{V-Nb} - \frac{N^2a}{V^2}. \tag{1}$$

Due to its mean-field character, the Van der Waals equation is not able to correctly describe the critical behavior of real fluids. However, it is possible to understand nontrivial fundamental properties of fluids with very modest theoretical effort.

1. Study the equation of state around the critical point. Use the fact that at $T = T_c$ the equation of state has an inflection point (why?), i.e.

   $$\left.\frac{\partial P}{\partial V}\right|_{T=T_c} = \left.\frac{\partial^2 P}{\partial V^2}\right|_{T=T_c} = 0. \tag{2}$$

   Determine the critical temperature T_c, the critical pressure P_c and critical volume V_c by expressing them in terms of the microscopic parameters a and b. Why is the result remarkable?

   Hint: Use the fact that the equation of state can be written as a cubic polynomical in V and that this polynomical must take the following form at the critical point (why?): $(V - V_c)^3 = 0$.

2. Show that the equation of state can be expressed in the form

   $$\left(P_R + \frac{3}{V_R^2}\right)(3V_R - 1) = 8T_R \tag{3}$$

   with $P_R = P/P_c$, $V_R = V/V_c$ and $T = T/T_c$. This relation is called the *law of corresponding states*. Discuss the significance of this law. What do you obtain for the ratio $P_cV_c/(k_BT_c)$?

3. Calculate the critical exponents of the liquid-gas phase transition and compare your results with the exponents of the nearest neighbor Ising model in one dimension:

   $$P - P_c \sim |V - V_c|^\delta, \quad \kappa_T = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T \sim |T - T_c|^{-\gamma}, \quad V_{\text{gas}} - V_{\text{liquid}} \sim |T - T_c|^\beta. \tag{4}$$

HINT: To obtain the exponent β expand the equation of state around V_c and T_c and apply the Maxwell construction for the coexistence region to determine the volumes V_{gas} and V_{liquid}. What's the underlying physics idea of the Maxwell construction?

Problem 4: Hubbard-Stratonovich transformation

The Hubbard-Stratonovich is a clever and very powerful method to handle interaction terms in many-body problems. It is used frequently in modern many-body methods, like quantum Monte-Carlo and lattice calculations. In this exercise we study an Ising model consisting of N spins with an infinite-range interaction. That means each spin $S_i = \pm 1$ interacts with every other spin in the system with strength J. The Hamiltonian is hence given by

$$H = -\frac{J}{2} \sum_{i,j=1}^{N} S_i S_j - B \sum_{i=1}^{N} S_i, \tag{5}$$

where B is a constant external magnetic field.

1. Which condition has to be fulfilled for the coupling strength J in order to make this model physically useful? Why is it useful to include a factor $\frac{1}{2}$ in the interaction term?

2. Show that the canonical partition sum only depends on the total spin $S_{\text{total}} = \sum_{i=1}^{N} S_i$ and write down the explicit form. Why is it not possible to factorize the partition sum into a product of single-particle partition sums?

3. Prove that
$$\exp\left[\frac{a}{N} x^2\right] = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{4\pi/(Na)}} \exp\left[-\frac{Na}{4} y^2 + axy\right]. \tag{6}$$
This is an example of a *Hubbard-Stratonovich transformation*. Use this result to show that the partition sum can be written in the form
$$Z = \int_{-\infty}^{\infty} dy \frac{e^{-NL(y)}}{\sqrt{2\pi/(N\beta\bar{J})}} \quad \text{with} \quad L(y) = \frac{\beta\bar{J}}{2} y^2 - \ln\left[2\cosh(\beta(B + \bar{J}y))\right]. \tag{7}$$
with $J = \bar{J}/N$. What have we achieved by applying the Hubbard-Stratonovich transformation Eq.(6)?

4. In the thermodynamic limit the partition sum can be evaluated using the method of steepest descents. This method is based on the fact that for $N \to \infty$ the integral is given by the maximum of the integrand at point $\bar{y}$:
$$\lim_{N\to\infty} Z \sim e^{-NL(\bar{y})}. \tag{8}$$
Explain why in the present case this methods leads to the exact solution and determine the equation satisfied by $\bar{y}$. Does this relation look familiar?

5. For the interpretation of the quantity $\bar{y}$ compute the magnetization per site: $M = -\frac{1}{N} \frac{\partial F}{\partial B}\Big|_{y=\bar{y}}$. Interpret the results and study the magnetization by investigating the implicit equation for $\bar{y}$ for $B = 0$. Determine the critical temperature and show that the critical exponent $\beta = 1/2$.

6. Show that the the magnetic susceptibility $\chi_T = \frac{\partial M}{\partial B}\Big|_T$ at $B = 0$ is given by the Curie-Weiss law:
$$\chi_T = \frac{(1 - M^2)}{T - T_c(1 - M^2)}, \quad \text{i.e.} \quad \gamma = 1. \tag{9}$$