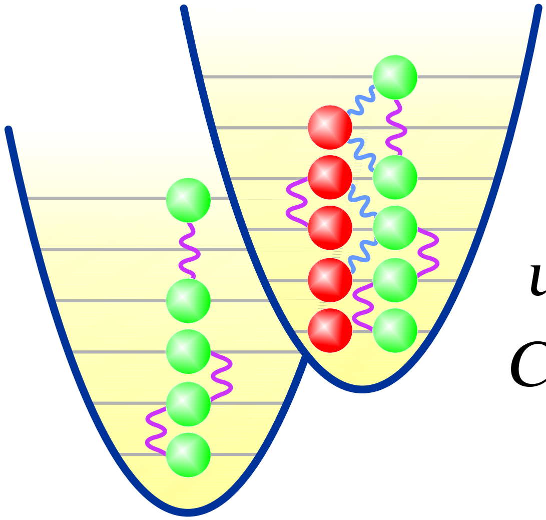


Structure and Stability of

Trapped Ultracold Fermi Gases

using Effective s- and p-Wave
Contact Interactions

R. Roth and H. Feldmeier
GSI, Darmstadt, Germany

Rauischholzhausen XII @ Rieneck; June 19-21, 2000

What is it all about?

Effective Contact Interaction

Inhomogeneous Fermi Gases in TFA

Stability against Collapse & Separation...

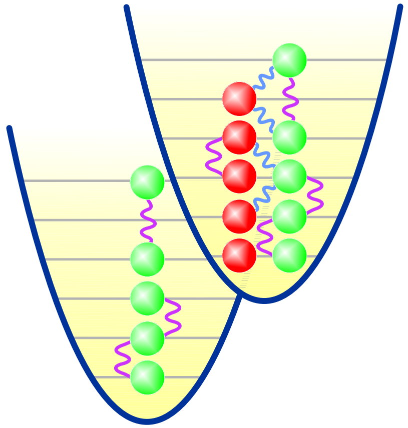
What Makes Its Attraction...

a macroscopic system which exhibits quantum properties

all relevant quantities are observable & tunable

size, density, particle number, mass, statistic, composition, temperature, distributions, **interaction strength**...

large composite bosons/fermions



meta stable many-body state

realization of a dilute Fermi gas

mean-field is appropriate

BE condensation
$T_{\text{BEC}} \sim \mu\text{K}$

BCS transition
$T_{\text{BCS}} \sim \text{nK}$

A Proper Effective Interaction...

Formal Requirements

- symmetries: translation, rotation, Galilean, parity, time reversal
- hermiticity

Physical Properties

- low density

$$\rho^{-1/3} \gg \text{range of interaction}$$

- low temperature

$$q^{-1} \gg \text{range of interaction}$$

Model Space

- mean-field type model
- model space does not contain two-body correlations

Relevant Observables

- static properties: groundstate structure, energy spectra, one-body densities,...
- dynamics: collective excitations, cooling processes, matter waves,...

Effective Interaction

- zero-range potential (for each partial wave)
- expectation value in two-body model-states equals the energy shift induced by the full interaction

Effective Contact Interaction I
Energy Shift & Phase Shifts

- ▶ consider a system of two particles interacting via a potential $v(r)$ of range λ with phase shifts $\eta_l(q)$

- relative two-body wave function for $r > \lambda$

$$R_{nl}(r) = A_{nl}\ j_l(q_{nl} r)$$
$$\bar{R}_{nl}(r) = \bar{A}_{nl}\big[j_l(\bar{q}_{nl} r) - \tan\eta_l(\bar{q}_{nl})\, n_l(\bar{q}_{nl} r)\big]$$

- boundary condition $R(\Lambda) = 0$

$$j_l(q_{nl}\Lambda) = 0$$
$$j_l(\bar{q}_{nl}\Lambda) = \tan\eta_l(\bar{q}_{nl})\ n_l(\bar{q}_{nl}\Lambda)$$

- asymptotic expansion of the Bessel & Neumann function ($q\Lambda \gg 1$); exact for s-wave!

$$q_{nl}\Lambda = \pi(n + \tfrac{l}{2})$$
$$\bar{q}_{nl}\Lambda = \pi(n + \tfrac{l}{2}) - [\eta_l(\bar{q}_{nl}) - \pi\, n_{\text{bound}}]$$

- momentum shift of the n-th positive energy state of the interacting spectrum with respect to the n-th free level

$$\Delta q_{nl}\Lambda = (\bar{q}_{nl} - q_{nl})\Lambda$$
$$= -[\eta_l(q_{nl}) - \pi\, n_{\text{bound}}] =: -\hat{\eta}_l(q_{nl})$$

- ▶ relative energy shift of the n-th positive energy level with respect to the n-th non-interacting level ($|\Delta q_{nl}/q_{nl}| \ll 1$)

$$\frac{\Delta E_{nl}}{E_{nl}} = -\frac{2}{q_{nl}\Lambda}\ \hat{\eta}_l(q_{nl})$$

Effective Contact Interaction II

Construction of the Interaction

- ansatz for the interaction operator of the l-th partial wave

$$\begin{aligned}\mathbf{v}_l^{\mathrm{ECI}} &= (\vec{\mathbf{q}}\,\vec{\mathbf{n}}_r)^l \; g_l \; \frac{\delta(r)}{4\pi\mathbf{r}^2} \; (\vec{\mathbf{n}}_r\vec{\mathbf{q}})^l \\ &= \int \mathrm{d}^3r \, |\vec{r}\rangle \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \; g_l \; \frac{\delta(r)}{4\pi r^2} \; \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \langle\vec{r}|\end{aligned}$$

- model space: free angular momentum eigenstates

$$\langle\vec{r}|nlm\rangle = R_{nl}(r)Y_{lm}(\Omega) \qquad R_{nl}(r) \overset{q_{nl}r\ll 1}{=} A_{nl}\,\frac{(q_{nl}r)^l}{(2l+1)!!}$$

- expectation value shall give the energy shift

$$\Delta E_{nl} \overset{!}{=} \langle nlm|\,\mathbf{v}_l^{\mathrm{ECI}}\,|nlm\rangle = \frac{g_l}{4\pi}\left[\frac{l!}{(2l+1)!!}\right]^2 A_{nl}^2\, q_{nl}^{2l}$$

- interaction strength as function of phase shifts

$$g_l = -\frac{4\pi}{2\mu}\left[\frac{(2l+1)!!}{l!}\right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}}$$

- parameterization of the phase shifts $\hat{\eta}_l(q)$ in terms of the scattering length a_l for $|qa_l| \ll 1$

$$g_l \approx \frac{4\pi}{2\mu}\,\frac{(2l+1)}{(l!)^2}\,a_l^{2l+1} + \mathcal{O}(q^2)$$

Summary

The Effective Contact Interaction

Conception

- hermitean effective contact interaction that reproduces the exact (low-energy) spectrum within a restricted model space that does not contain two-body correlations
- assuming low density & temperature: $\rho^{-1/3}$ and q^{-1} large compared to the interaction range

Realization

$$\langle \vec{r} | \, \mathbf{v}_l^{\mathrm{ECI}} \, | \vec{r} \rangle = \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \; g_l \; \frac{\delta(r)}{4\pi r^2} \; \frac{\overrightarrow{\partial}^l}{\partial r^l}$$

$$g_l = -\frac{4\pi}{m} \left[\frac{(2l+1)!!}{l!} \right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}} \approx \frac{4\pi}{m} \; \frac{(2l+1)}{(l!)^2} \; a_l^{2l+1}$$

Application

- energy-density functional for fermionic (bosonic) matter including p-wave interactions
- Thomas-Fermi approximation for trapped ultracold Fermi (Bose) gases, exploration of static properties
- dynamical description of cooling, collective dynamics, vortex dynamics...

A Model for

Trapped Fermi Gases

❶ *Homogeneous Fermi Gas*

- homogeneous gas of Ξ distinguishable fermionic components interacting via the s- and p-wave part of the ECI

$$\mathbf{H} = \mathbf{T} + \mathbf{V}_0^{\text{ECI}} + \mathbf{V}_1^{\text{ECI}}$$

- mean-field groundstate $\left|\Psi_{\text{MF}}\right\rangle$ is the antisymmetrized product of all plane-wave states with momenta up to the Fermi momentum κ_ξ for each component

$$\mathcal{E}_{\text{hom}}(\kappa_1, ..., \kappa_\Xi) = \left\langle\Psi_{\text{MF}}\right| \mathbf{H} \left|\Psi_{\text{MF}}\right\rangle / V$$

❷ *Thomas-Fermi Approximation*

- energy-density of the inhomogeneous system is given locally by $\mathcal{E}_{\text{hom}}$ plus contributions of external fields

$$\mathcal{E}[\kappa_1, ..., \kappa_\Xi](\vec{x}) \overset{\text{TFA}}{=} \mathcal{E}_{\text{hom}}(\kappa_1(\vec{x}), ..., \kappa_\Xi(\vec{x})) + \frac{1}{6\pi^2} \sum_\xi U_\xi(\vec{x})\, \kappa_\xi^3(\vec{x})$$

❸ *Functional Variation*

- the one-body density that minimizes the energy for a given particle number describes the groundstate of the system

$$\frac{\delta}{\delta\kappa_\xi} \int \mathrm{d}^3x \left(\mathcal{E}[\kappa_1, ..., \kappa_\Xi](\vec{x}) - \frac{1}{6\pi^2} \sum_{\xi'} \mu_{\xi'} \kappa_{\xi'}^3(\vec{x}) \right) = 0$$

► *extremum condition*: derivative of the integrand with respect to $\kappa_\xi(\vec{x})$ has to vanish for each point $\vec{x}$

Energy-Density for Trapped Fermions

One-Component System

$$\mathcal{E}[\kappa](\vec{x}) =$$

$$= \frac{1}{6\pi^2}\, U(\vec{x})\, \kappa^3(\vec{x})$$ — trap

$$+ \frac{1}{20\pi^2 m}\, \kappa^5(\vec{x})$$ — kinetic

✗ — s-wave

$$+ \frac{a_1^3}{30\pi^3 m}\, \kappa^8(\vec{x})$$ — p-wave

- s-wave contact interaction does not contribute due to the Pauli principle
- leading interaction term is p-wave

Two-Component System

$$\mathcal{E}[\kappa_1, \kappa_2](\vec{x}) =$$

$$= \frac{1}{6\pi^2}\, [U_1(\vec{x})\, \kappa_1^3(\vec{x}) + U_2(\vec{x})\, \kappa_2^3(\vec{x})]$$ — trap

$$+ \frac{1}{20\pi^2 m}\, [\kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x})]$$ — kinetic

$$+ \frac{a_0}{9\pi^3 m}\, \kappa_1^3(\vec{x})\kappa_2^3(\vec{x})$$ — s-wave

$$+ \frac{a_1^3}{30\pi^3 m}\, [\kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) + + \tfrac{1}{2}\kappa_1^3(\vec{x})\kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\kappa_2^3(\vec{x})]$$ — p-wave

- s-wave interaction between particles belonging to different components
- p-wave interaction within and between components

Effect of p-Wave Interactions

TF Groundstate Solution

- energy-density

$$\mathcal{E}[\kappa](\vec{x}) = \frac{1}{6\pi^2} U(\vec{x}) \kappa^3(\vec{x}) + \frac{1}{20\pi^2 m} \kappa^5(\vec{x}) + \frac{a_1^3}{30\pi^3 m} \kappa^8(\vec{x})$$

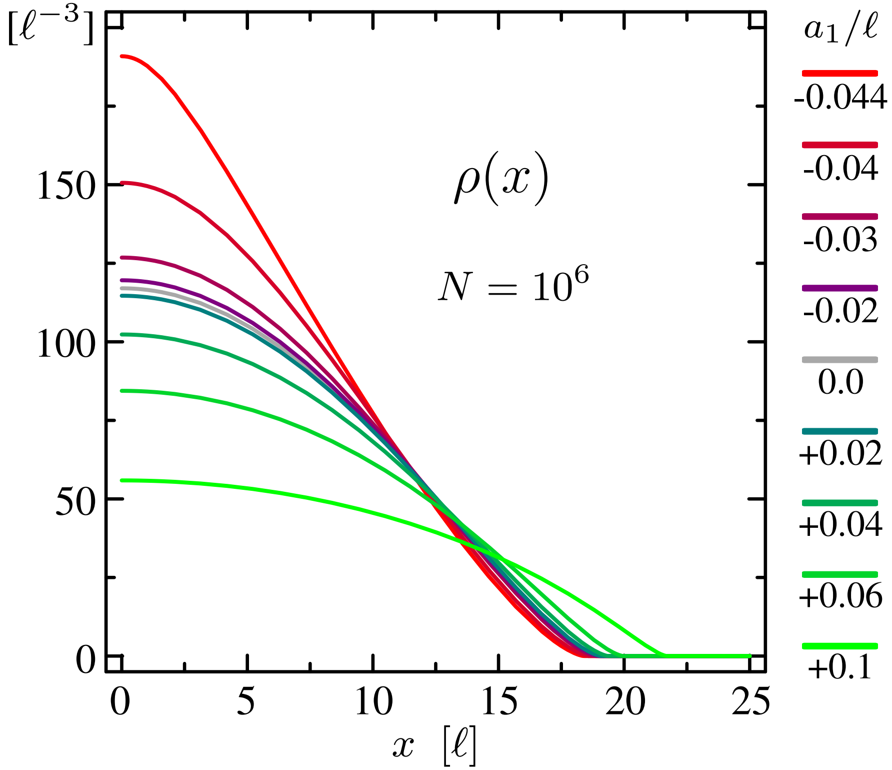
- functional variation gives extremum condition

$$m[\mu - U(\vec{x})] = \frac{1}{2}\kappa^2(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

- harmonic trap (spherical)

$$U(\vec{x}) = \frac{x^2}{2m\ell^4} \qquad \ell = \frac{1}{\sqrt{m\omega}}$$

▶ groundstate density $\rho(\vec{x})$ is obtained by point-wise solution of the extremum condition for given $\mu(N), a_1/\ell$



example: $\left.\begin{aligned} \ell &= 1\mu\text{m} \\ a_1 &= 200 a_\text{B} \end{aligned}\right\} \rightarrow a_1/\ell \approx 0.01$

p-Wave Attraction & Stability

Extremum Condition

$$m[\mu - U(\vec{x})] = \frac{1}{2}\kappa^2(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

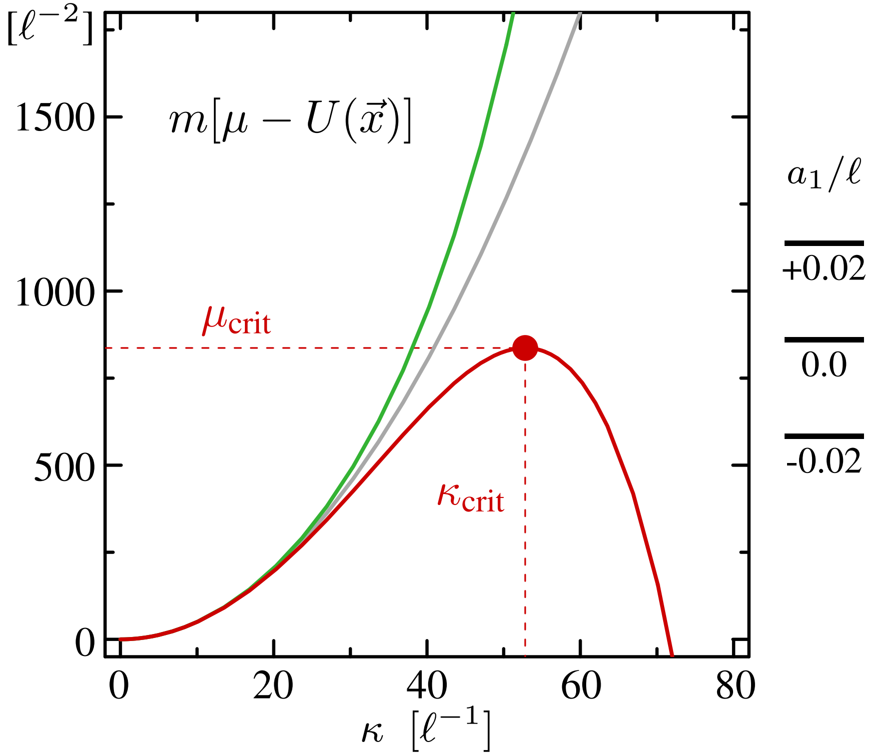
- for attractive p-wave interactions the r.h.s. exhibits a maximum at $(\kappa_{\text{crit}}, \mu_{\text{crit}})$

Stability Conditions

$$\kappa_{\text{crit}} = \frac{(3\pi)^{1/3}}{2\,|a_1|} \qquad \rho_{\text{crit}} = \frac{1}{16\pi\,|a_1|^3}$$

$$\mu_{\text{crit}} = \frac{3(3\pi)^{3/2}}{40m\,|a_1|^2} \qquad N_{\text{crit}} = \frac{(0.445\,\ell)^6}{|a_1|^6}$$

▶ there is no stable mean-field solution for densities or particle numbers beyond the critical values



a_1 $[a_B]$	κ_{crit} $[\mu\text{m}^{-1}]$	ρ_{crit} $[\mu\text{m}^{-3}]$	$m\mu_{\text{crit}}$ $[\mu\text{m}^{-2}]$	N_{crit}
-200	106	20000	22000	7.8×10^9
-2000	10.6	20	220	7800

$\ell = 1\mu\text{m}$

Extremum Condition & Stability

Extremum Condition

- assume same $[\mu - U(\vec{x})]$ and same $\kappa(\vec{x})$ for both components

$$m[\mu - U(\vec{x})] = \\ = \frac{1}{2}\kappa^2(\vec{x}) + \frac{2\,a_0}{3\pi}\kappa^3(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

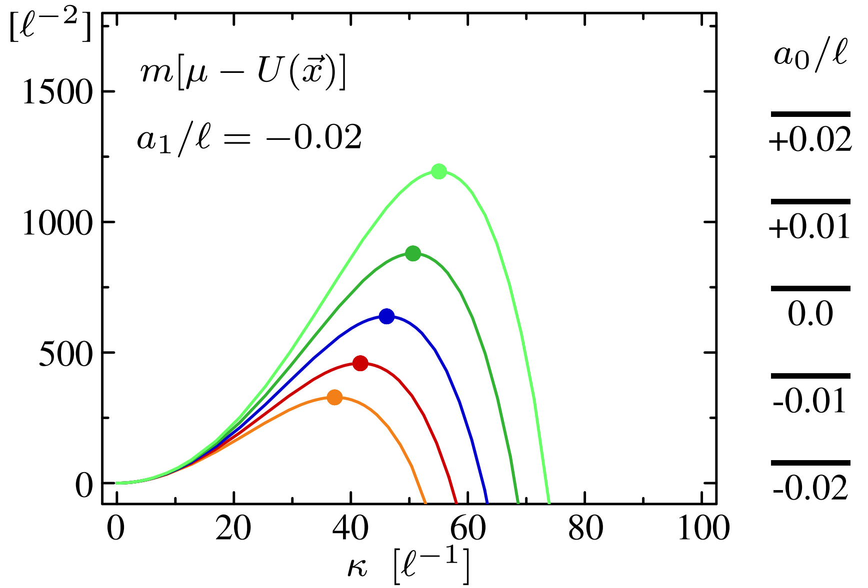
Stability Conditions

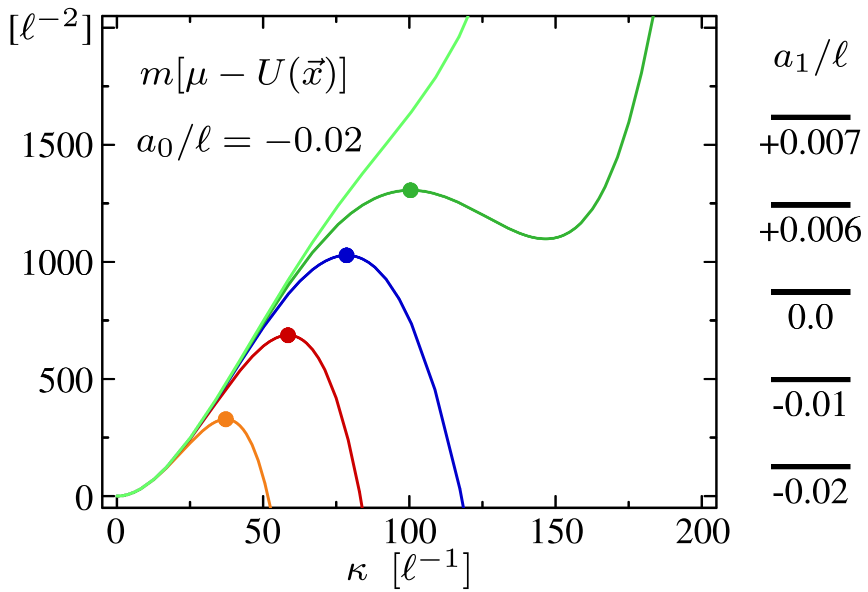
$$-a_0\,\kappa_{\text{crit}} - 2(a_1\,\kappa_{\text{crit}})^3 = \frac{\pi}{2}$$

- collapse may occur if one of the scattering lengths is negative
- p-wave repulsion prevents collapse due to s-wave attraction if

$$\frac{a_1}{|a_0|} > \frac{2}{3\pi^{2/3}} \approx 0.31$$

- even a strong s-wave repulsion can not stabilize the collapse due to p-wave attraction





Two-Component Fermi Gas

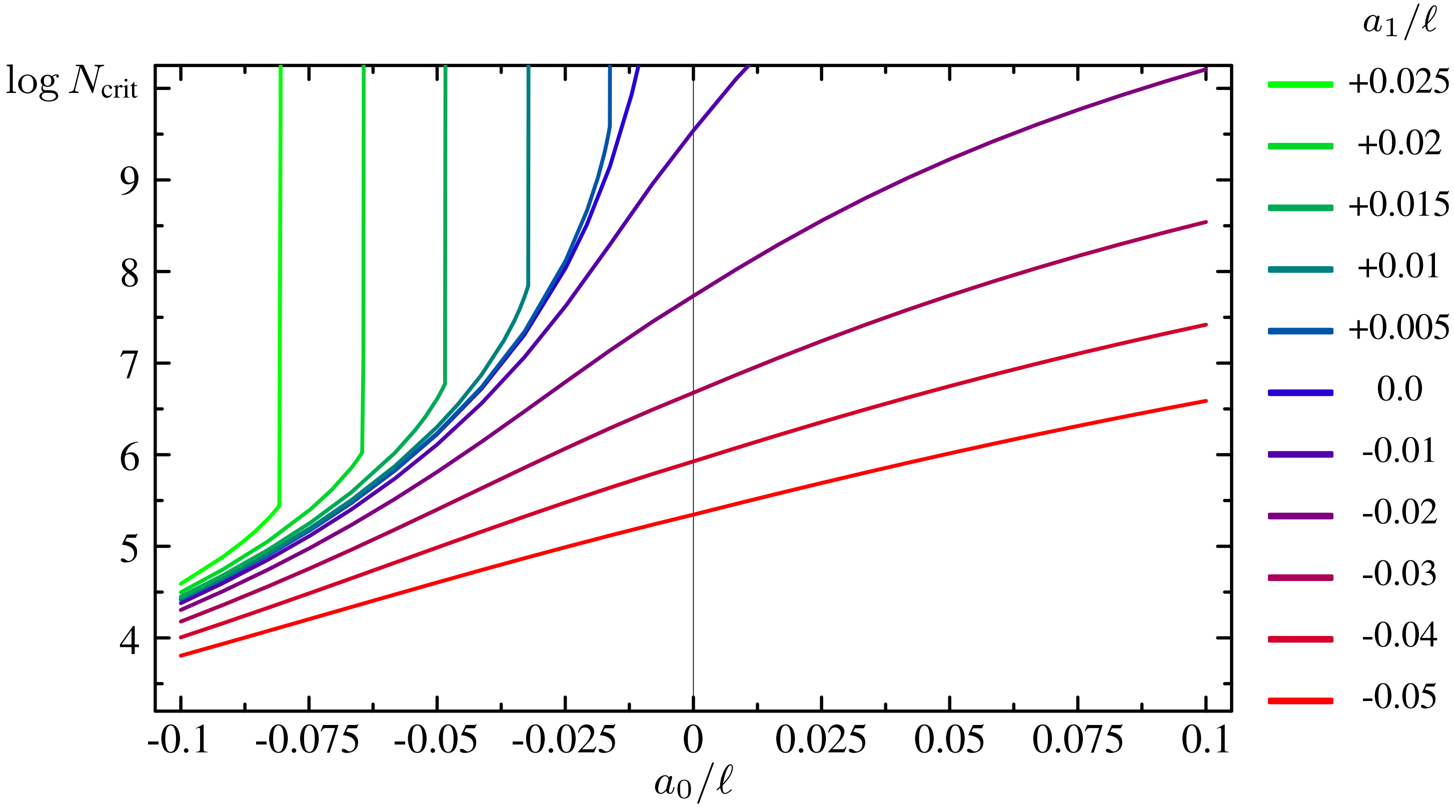
Critical Particle Number for Collapse

- assume harmonic trap with average oscillator length $\ell = (\ell_x^2 \ell_y^2 \ell_z^2)^{1/6}$
- solve extremum condition for the critical chemical potential and calculate N_{crit}

absolute stabilization due to p-wave repulsion $a_1/|a_0| > 2/(3\pi^{2/3})$

p-wave attraction lowers critical particle number substantially

collapse due to p-wave attraction is not stabilized by s-wave repulsion

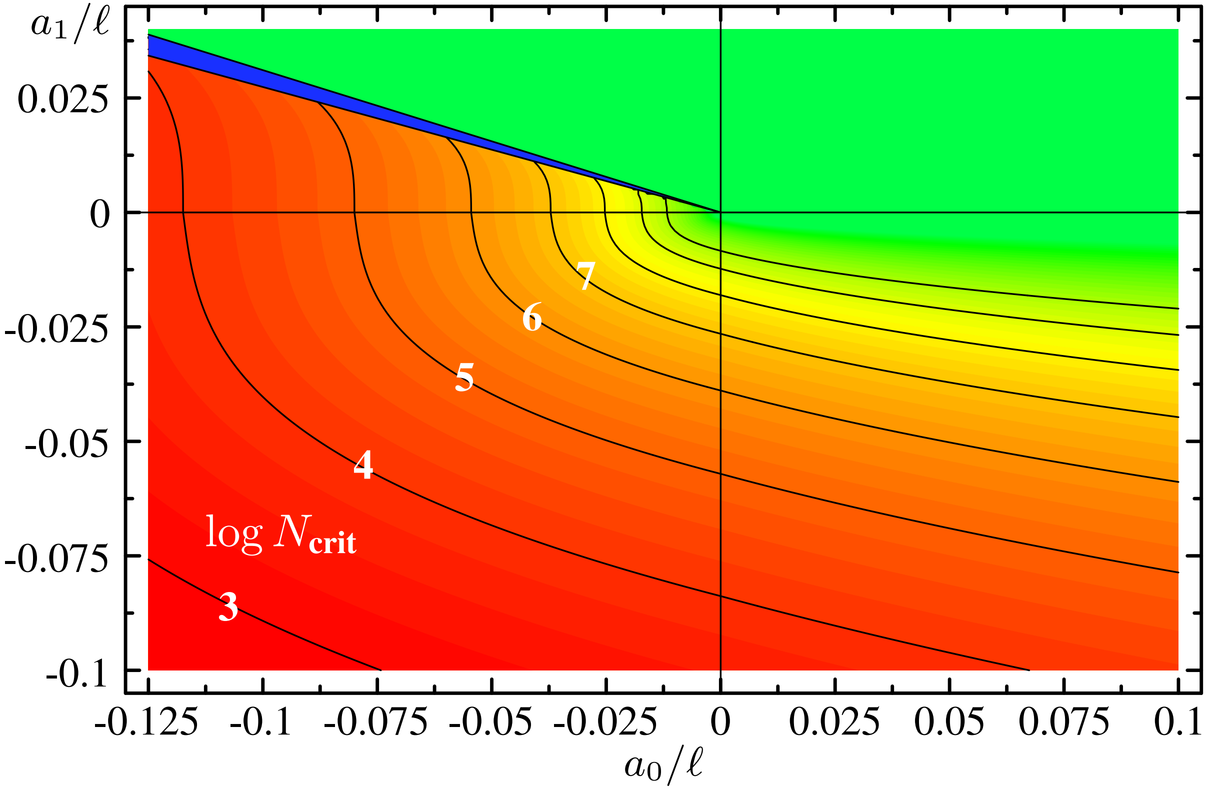


Two-Component Fermi Gas

Stability Map

p-wave stabilized high-density phase

metastable condensate for any particle number



limited particle number for the metastable condensate

Two-Component Fermi Gas

p-Wave Stabilized High-Density Phase

Extremum Condition: $a_0 < 0,\ a_1 > 0$

$$a_1/|a_0| < \frac{2}{3\pi^{2/3}} \approx 0.31$$

➠ r.h.s. of the extremum condition shows separated low- and high-density branches

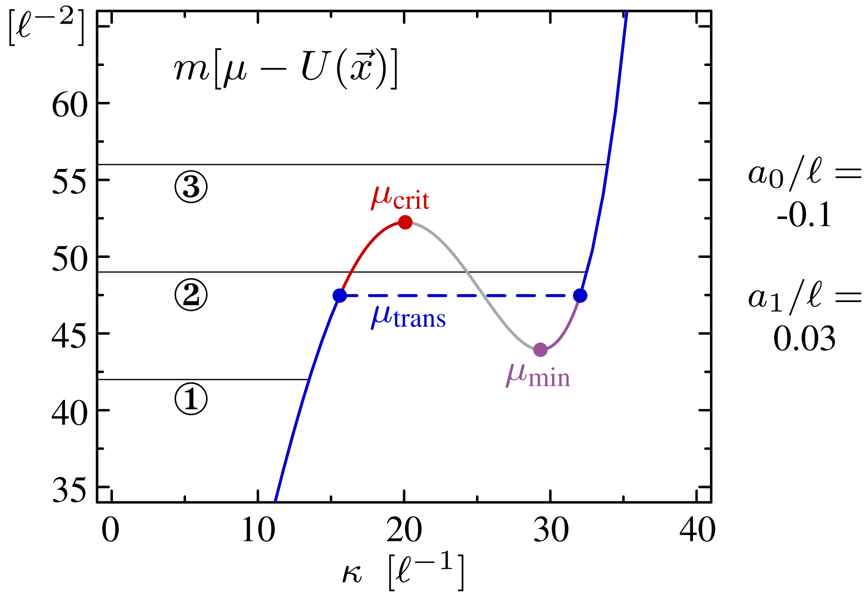
$$a_1/|a_0| > \sqrt[3]{\frac{160}{729\pi^2}} \approx 0.28$$

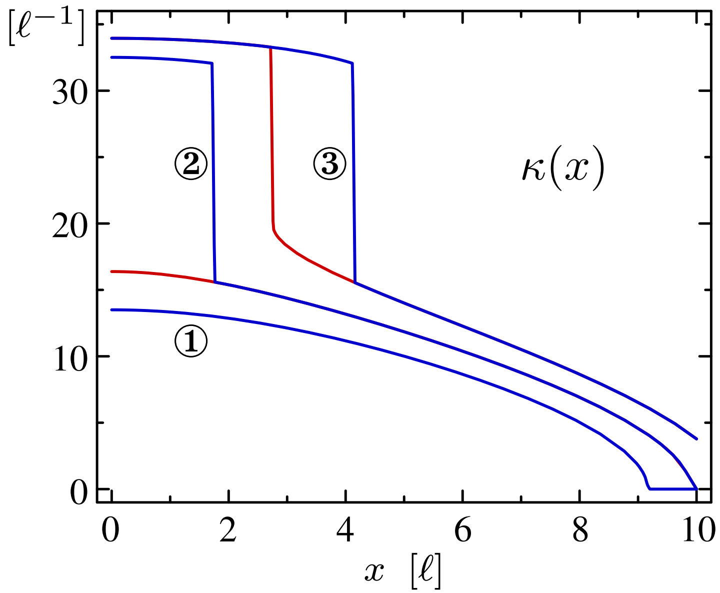
➠ no self-bound solutions on the high-density branch ($\mu_{\text{min}} > 0$)

Solution Structure

$[\mu - U(\vec{x})]$

$< \mu_{\text{min}}$	only low-density solution exists
μ_{min} ... μ_{trans}	low- and high-density solution exist; low-density is energetically favored
μ_{trans} ... μ_{crit}	both solutions exist; high-density is energetically favored, low-density is metastable
$> \mu_{\text{crit}}$	only high-density solution exists



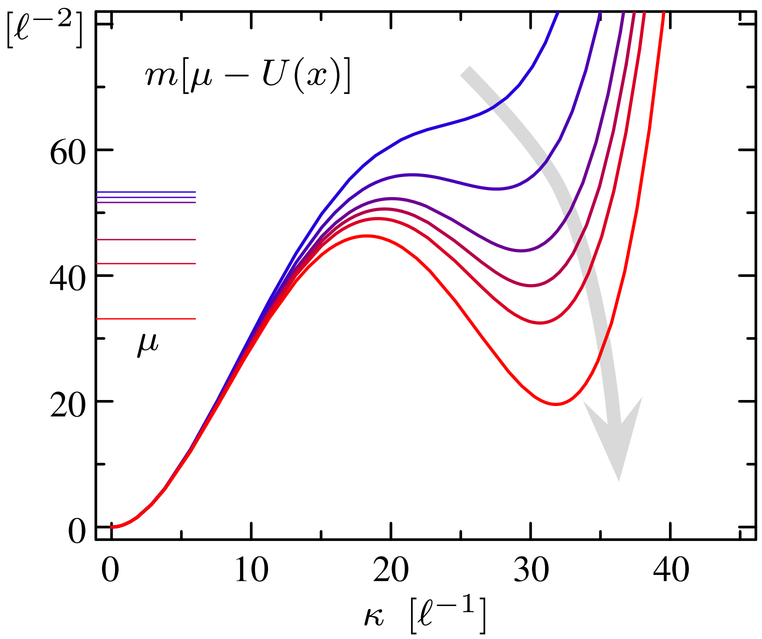


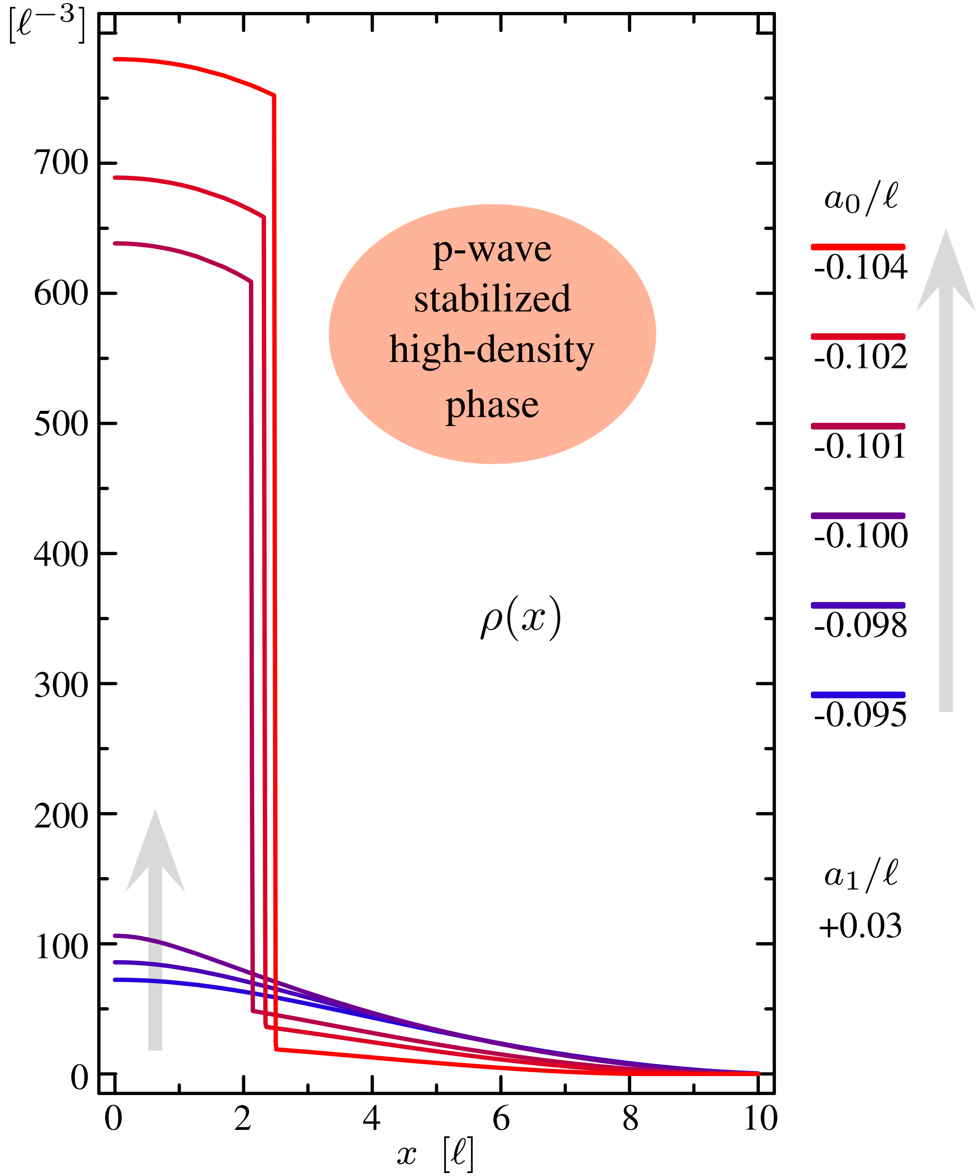
High-Density Phase: “Gedankenexperiment”

“Experimental Setup”

- two-component Fermi gas in a spherical trap with oscillator length ℓ
- fixed particle number $N_1 = N_2 = 60000$

➠ increase the strength of the attractive s-wave interaction adiabatically and keep the repulsive p-wave fixed...





Component Separation

Energy-Density

$$\mathcal{E}[\kappa_1,\kappa_2](\vec{x}) = \frac{1}{6\pi^2}[U_1(\vec{x})\,\kappa_1^3(\vec{x}) + U_2(\vec{x})\,\kappa_2^3(\vec{x})] + \frac{1}{20\pi^2 m}[\kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x})] + \frac{a_0}{9\pi^3 m}\kappa_1^3(\vec{x})\kappa_2^3(\vec{x}) + \frac{a_1^3}{30\pi^3 m}[\kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) + \tfrac{1}{2}\kappa_1^3(\vec{x})\kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\kappa_2^3(\vec{x})]$$

- for repulsive s-wave (p-wave) interactions it may be energetically favorable to separate both components spatially

Overlapping Configuration

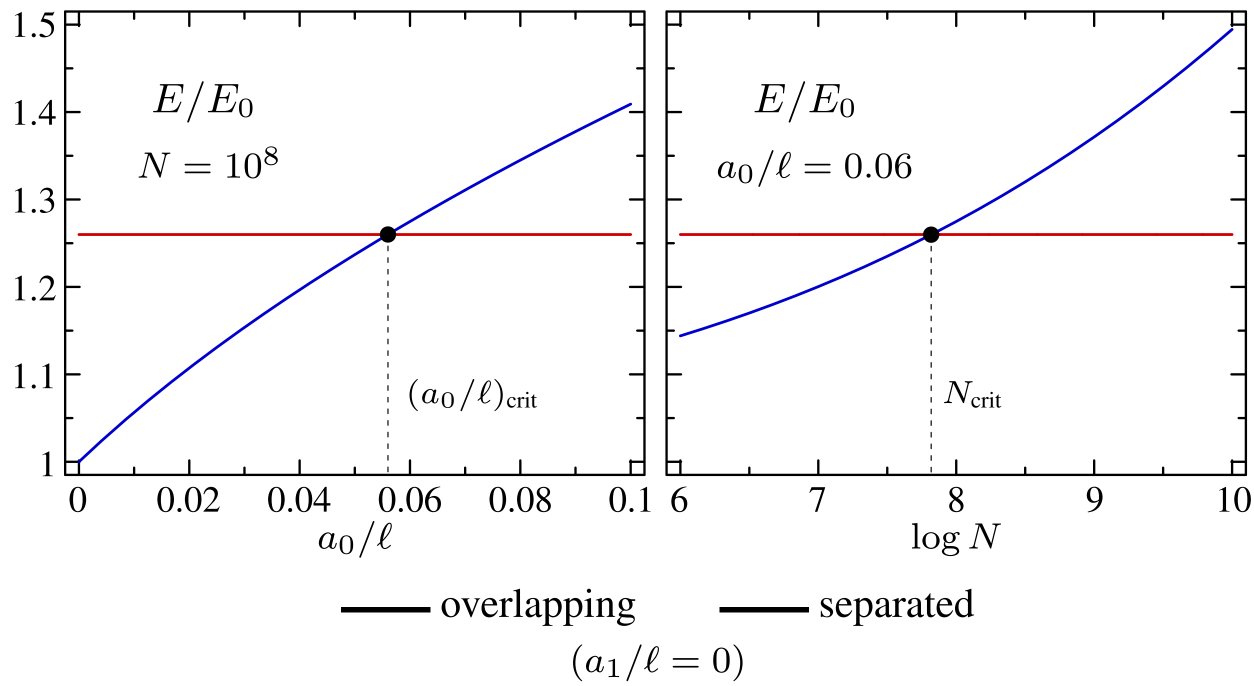
$$\kappa_1(\vec{x}) \equiv \kappa_2(\vec{x})$$

➠ s- and p-wave interaction terms contribute

Separated Configuration

$$\kappa_1(\vec{x})\kappa_2(\vec{x}) \equiv 0$$

➠ s-wave interaction does not contribute at all



Two-Component Fermi Gas

Critical Particle Number for Separation

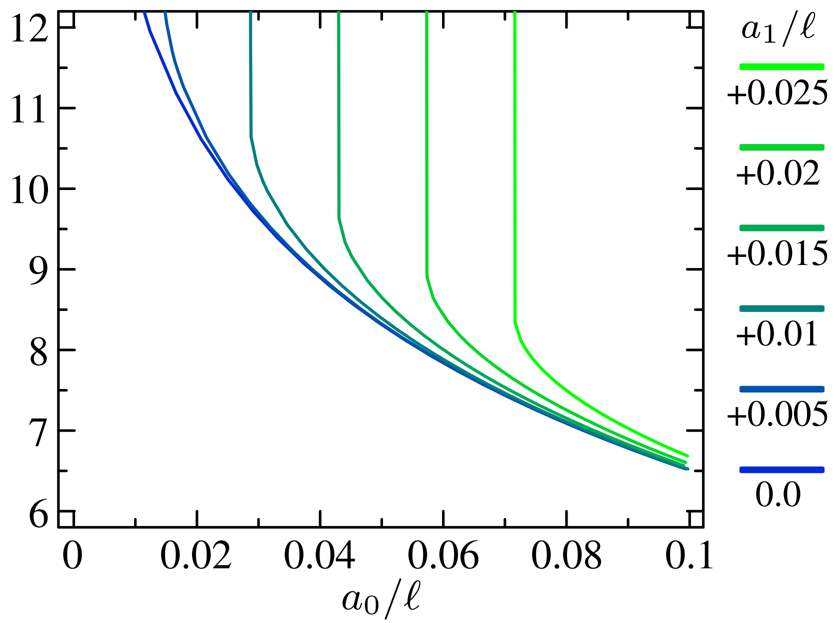
- assume harmonic trap with average oscillator length $\ell = (\ell_x^2 \ell_y^2 \ell_z^2)^{1/6}$
- choose μ such that the particle numbers and the energy of the overlapping and the separated configuration are equal

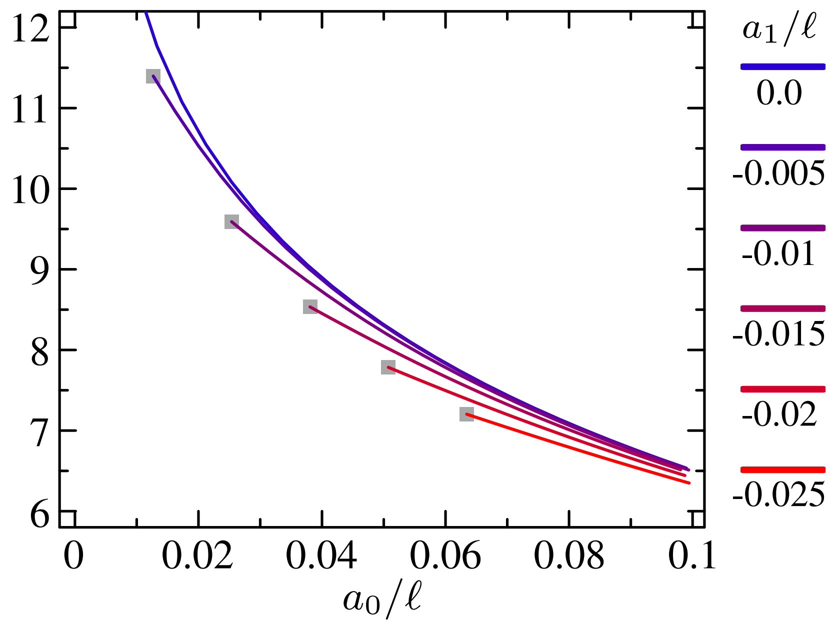
p-wave repulsion raises critical particle number

absolute stabilization due to p-wave repulsion $a_1/a_0 > 0.349$

p-wave attraction lowers critical particle number

collapse of the separated phase due to p-wave attraction

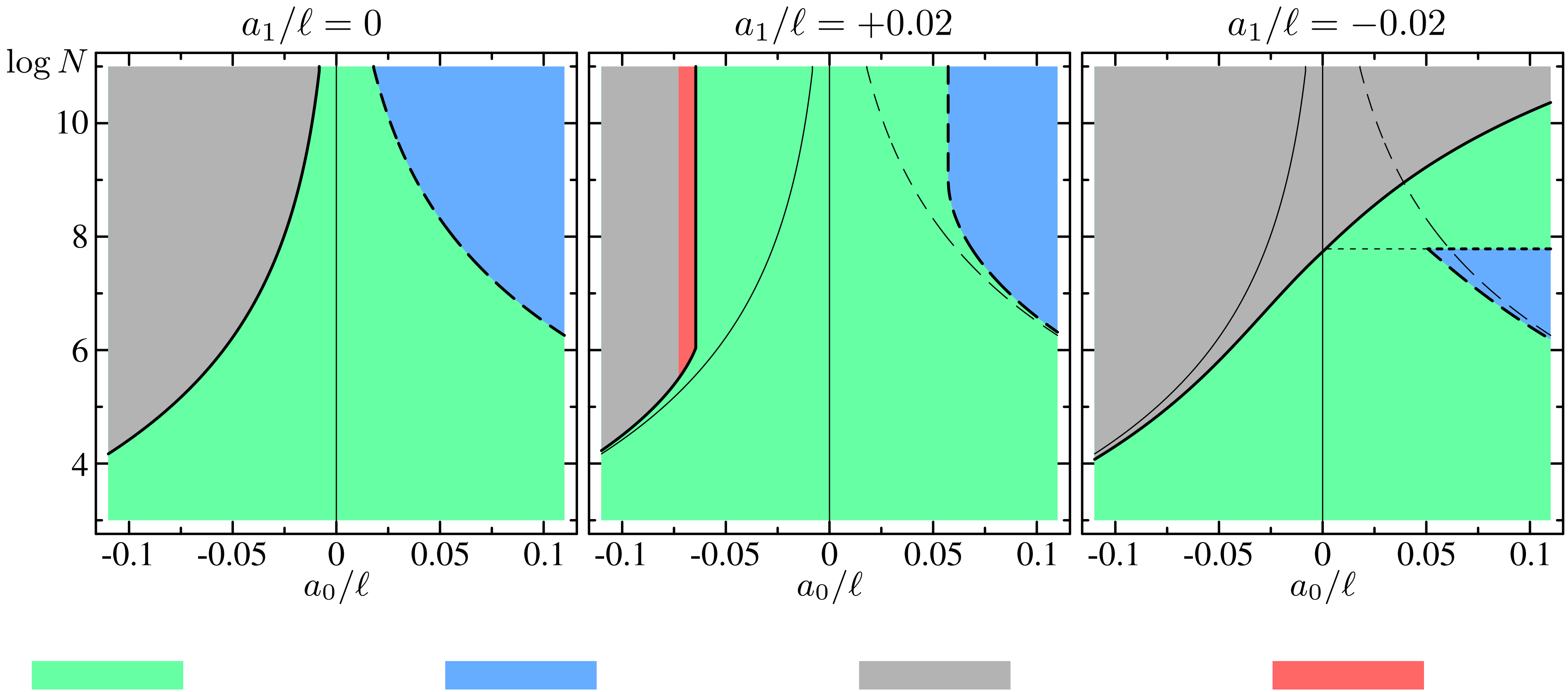




R. Roth - 06/2000

Two-Component Fermi Gas

“Phase Diagrams”



metastable state with identical overlapping density profiles

metastable state with spatially separated components

unstable against collapse to a high-density state

p-wave stabilized high-density phase in the center of the trap

Summary

Procedure

- derived an Effective Contact Interaction that reproduces the two-body spectrum
- used s- and p-wave terms to set up the energy-density of an inhomogeneous Fermi gas in Thomas-Fermi approximation
- investigated the effects of s- and p-wave interactions on the structure and stability of one- and two-component systems

Results

- particle number/density of the one-component system is limited for attractive p-wave interactions
- complex interplay between s- and p-wave interactions in the two-component system concerning collapse and component separation
- novel phenomena: absolute stabilization due to p-wave repulsion, p-wave stabilized high-density phase
- **do not neglect p-wave interactions from the outset!**