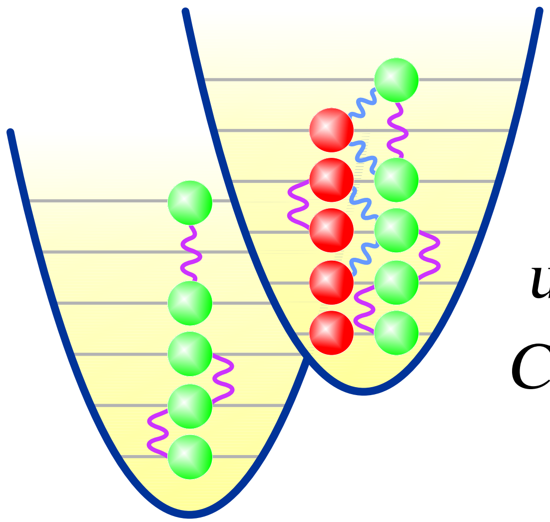


*Structure and Stability of*

# *Trapped Ultracold Fermi Gases*

*using Effective s- and p-Wave Contact Interactions*


***R. Roth*** *and H. Feldmeier*

*Gesellschaft für Schwerionenforschung, Darmstadt, Germany*


What is it all about?

Effective Contact Interaction

Inhomogeneous Fermi Gases in TFA

Stability against Collapse & Separation...

Science 285 (1999) 1703, 10 Sept. 1999

# Onset of Fermi Degeneracy in a Trapped Atomic Gas

B. DeMarco and D. S. Jin*†

An evaporative cooling strategy that uses a two-component Fermi gas was employed to cool a magnetically trapped gas of 7 × $10^5$ $^{40}$K atoms to 0.5 of the Fermi temperature $T_F$. In this temperature regime, where the state occupation at the lowest energies has increased from essentially zero at high temperatures to nearly 60 percent, quantum degeneracy was observed as a barrier to evaporative cooling and as a modification of the thermodynamics. Measurements of the momentum distrib... energy of the confined Fermi gas directly reveal...

cooling a cloud of neutral $^{40}$K atoms kept in a magnetic trap

$N \approx 10^5 \ldots 10^6$

$T \approx 0.3\,\mu\text{K}$
$\approx 0.5\,\varepsilon_F$

$\ell \approx 1\,\mu\text{m}$

$\rho \approx 10\,\mu\text{m}^{-3}$

$\tau \approx 300\text{s}$

## *Some Facts*

- $^{40}$K has fractional total spin: fermion

$$F = \underset{I}{4} \pm \underset{S}{1/2} = \tfrac{9}{2}, \tfrac{7}{2}$$

- magnetic trap: inhomogeneous external magnetic field couples to the magnetic moment of the atoms and builds a trap
- two selected substates are kept in the trap: $\left|F{=}\tfrac{9}{2}, m_F{=}\tfrac{9}{2}\right\rangle$ & $\left|F{=}\tfrac{9}{2}, m_F{=}\tfrac{7}{2}\right\rangle$
- evaporative cooling: system is cooled by selective removal of high-energy atoms and re-thermalization
- life-time is limited due to collisions with residual room-temperature atoms
- system can be imaged by resonance absorption of light (destructive) or phase-contrast techniques (nearly non-destructive)

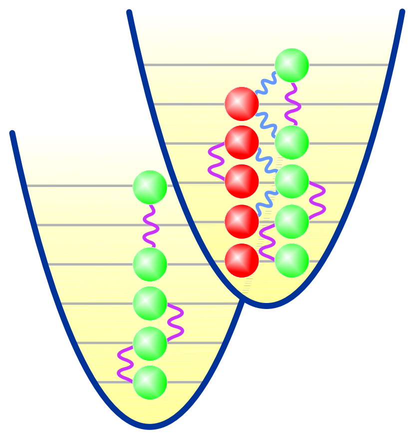
# *What Makes It So Attractive...*

a macroscopic system which exhibits quantum properties

*all* relevant quantities are observable & tunable

size, density, particle number, mass, statistic, composition, temperature, distributions, **interaction strength**...

large composite bosons/fermions



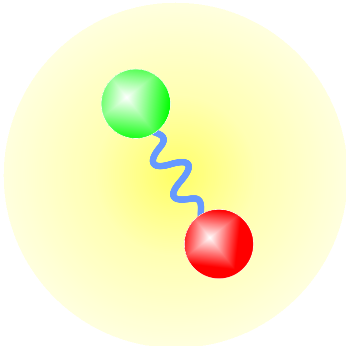
meta stable many-body state

realization of a dilute Fermi gas

mean-field is appropriate

BE condensation
$T_{\mathrm{BEC}} \sim \mu\mathrm{K}$

BCS transition
$T_{\mathrm{BCS}} \sim \mathrm{nK}$



# *Constructing a Proper Effective Interaction*

- Short-Range Correlations
- Proper Effective Interaction for Ultracold Dilute Gases
- Effective Contact Interaction

*Why Effective Interactions?*

# *The Problem: Short-Range Correlations*

## *Interaction*

many realistic two-body interactions show a strong short-range repulsion

(e.g. nucleon-nucleon & van der Waals interactions)

## *Correlations*

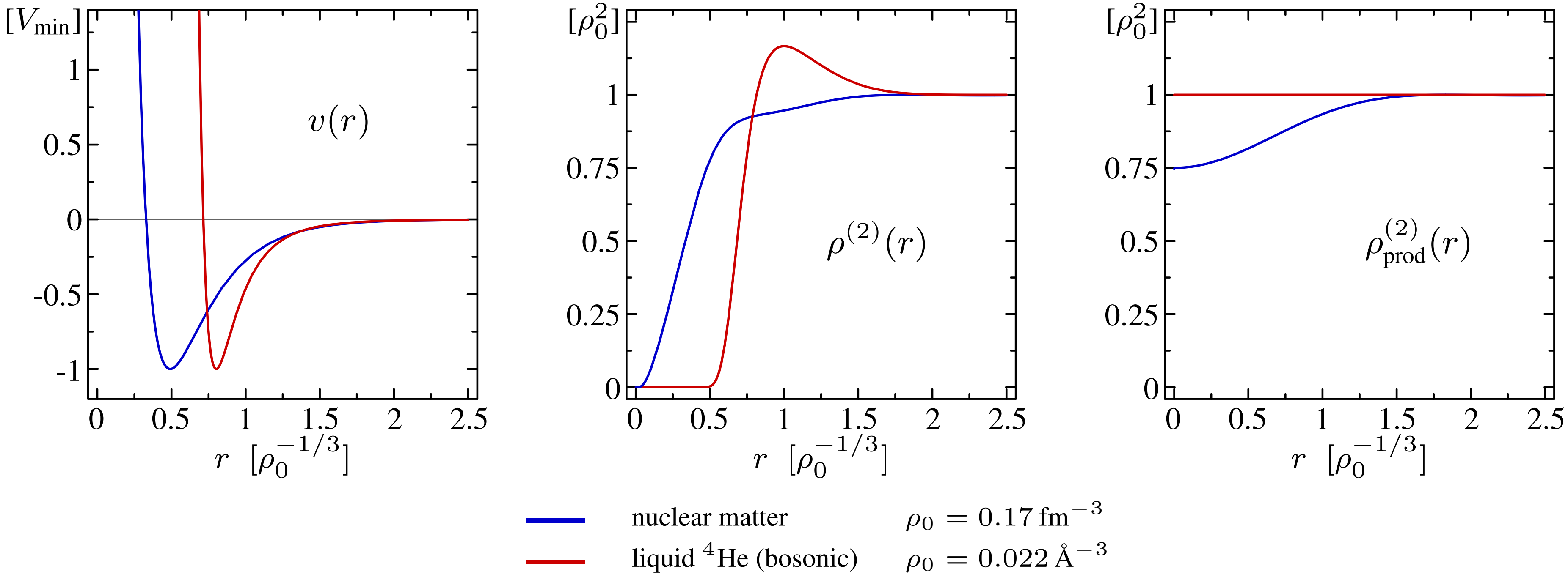
core induces strong short-range correlations in many-body state

(e.g. correlation hole in two-body density)

## *Product States*

short-range correlations cannot be described by product-type states

(e.g. mean-field, superposition of few product states,...)



*Why Effective Interactions?*

# *The Problem: Short-Range Correlations*

### *Interaction*

many realistic two-body interactions show a strong short-range repulsion

(e.g. nucleon-nucleon & van der Waals interactions)

### *Correlations*

core induces strong short-range correlations in many-body state

(e.g. correlation hole in two-body density)

### *Product States*

short-range correlations cannot be described by product-type states

(e.g. mean-field, superposition of few product states,...)

### *Effective Interaction*

replace the full potential by a tamed effective interaction

### *Correlated States*

include correlations in many-body model-space

### *Effective Contact Interaction ECI*

# *A Proper Effective Interaction...*

system is very dilute and cold

$\rho^{-1/3} \gg$ range of interaction

$q^{-1} \gg$ range of interaction

treat the many-body problem in a restricted model-space that does not contain correlations

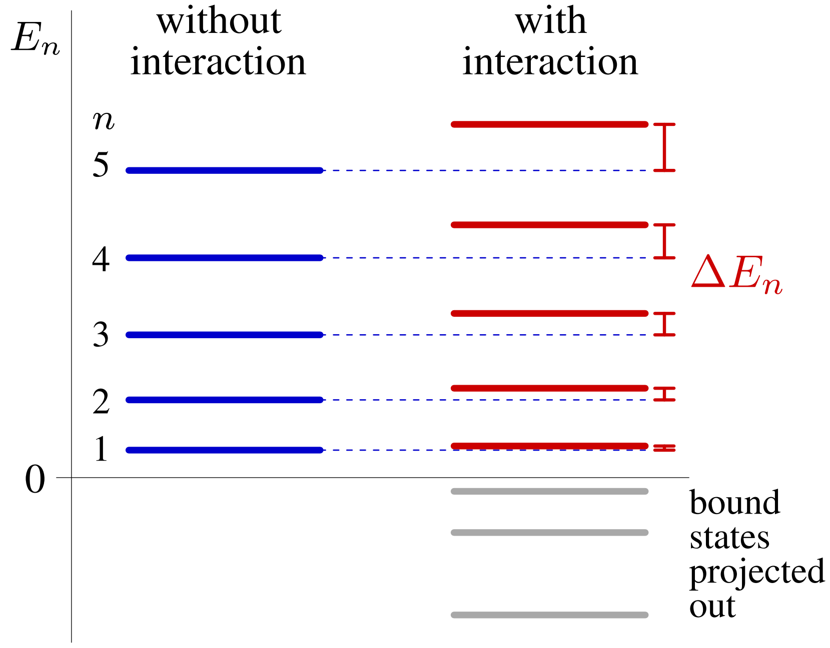
looking for the structure of non-selfbound states in an external potential

hermitean interaction operator that obeys standard symmetries (e.g. translation, rotation,...)

## *Effective Contact Interaction*

- zero-range potential (for each partial wave)
- expectation value in two-body model-states equals the energy shift induced by the full interaction

$$\langle \phi_n^{\text{model}} | \mathbf{v}^{\text{ECI}} | \phi_n^{\text{model}} \rangle \stackrel{!}{=} \Delta E_n$$



## *Effective Contact Interaction I*
# *Energy Shift & Phase Shifts*

- consider a system of two particles interacting via a potential $v(r)$ of range $\lambda$ with phase shifts $\eta_l(q)$

- relative two-body wave function for $r > \lambda$

$$R_{nl}(r) = A_{nl}\ j_l(q_{nl} r)$$
$$\bar{R}_{nl}(r) = \bar{A}_{nl}\big[j_l(\bar{q}_{nl} r) - \tan\eta_l(\bar{q}_{nl})\, n_l(\bar{q}_{nl} r)\big]$$

- auxiliary boundary condition $R_{nl}(\Lambda) = 0$

$$j_l(q_{nl}\Lambda) = 0$$
$$j_l(\bar{q}_{nl}\Lambda) = \tan\eta_l(\bar{q}_{nl})\ n_l(\bar{q}_{nl}\Lambda)$$

- asymptotic expansion of the Bessel & Neumann function ($q\Lambda \gg l$); exact for s-wave!

$$q_{nl}\Lambda = \pi(n + \tfrac{l}{2})$$
$$\bar{q}_{nl}\Lambda = \pi(n + \tfrac{l}{2}) - \big[\eta_l(\bar{q}_{nl}) - \pi\, n_{\text{bound}}\big]$$

- momentum shift of the $n$-th positive energy state of the interacting spectrum with respect to the $n$-th free level

$$\Delta q_{nl}\Lambda = (\bar{q}_{nl} - q_{nl})\Lambda$$
$$= -\big[\eta_l(q_{nl}) - \pi\, n_{\text{bound}}\big] =: -\hat{\eta}_l(q_{nl})$$

- relative energy shift of the $n$-th positive energy level with respect to the $n$-th non-interacting level ($|\Delta q_{nl}/q_{nl}| \ll 1$)

$$\frac{\Delta E_{nl}}{E_{nl}} = -\frac{2}{q_{nl}\Lambda}\ \hat{\eta}_l(q_{nl})$$

# *Effective Contact Interaction II*
# *Construction of the Interaction*

▶ ansatz for the interaction operator of the $l$-th partial wave

$$
\begin{aligned}
\mathbf{v}_l^{\mathrm{ECI}} &= (\vec{\mathbf{q}}\,\vec{\mathbf{n}}_r)^l \; g_l \; \frac{\delta(r)}{4\pi\mathbf{r}^2} \; (\vec{\mathbf{n}}_r\vec{\mathbf{q}})^l \\
&= \int \mathrm{d}^3r \, |\vec{r}\rangle \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \; g_l \; \frac{\delta(r)}{4\pi r^2} \; \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \langle\vec{r}|
\end{aligned}
$$

• model space: free angular momentum eigenstates

$$
\langle\vec{r}|nlm\rangle = R_{nl}(r) Y_{lm}(\Omega) \qquad R_{nl}(r) \overset{q_{nl}r \ll 1}{=} A_{nl} \frac{(q_{nl}r)^l}{(2l+1)!!}
$$

• expectation value shall give the energy shift

$$
\Delta E_{nl} \overset{!}{=} \langle nlm| \, \mathbf{v}_l^{\mathrm{ECI}} \, |nlm\rangle = \frac{g_l}{4\pi} \left[\frac{l!}{(2l+1)!!}\right]^2 A_{nl}^2 \, q_{nl}^{2l}
$$

▶ interaction strength as function of phase shifts

$$
g_l = -\frac{4\pi}{2\mu} \left[\frac{(2l+1)!!}{l!}\right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}}
$$

▶ parameterization of the phase shifts $\hat{\eta}_l(q)$ in terms of the scattering length $a_l$ for $|q\, a_l| \ll 1$

$$
g_l \approx \frac{4\pi}{2\mu} \frac{(2l+1)}{(l!)^2} {a_l}^{2l+1} \;+\; \mathcal{O}(q^2)
$$

# *The Effective Contact Interaction*

*Conception*

- hermitean effective contact interaction that reproduces the exact (low-energy) spectrum within a restricted model space that does not contain two-body correlations
- assuming low density & temperature: $\rho^{-1/3}$ and $q^{-1}$ large compared to the interaction range

*Realization*

$$\mathbf{v}_l^{\mathrm{ECI}} = \int \mathrm{d}^3 r \; |\vec{r}\rangle \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \, g_l \, \frac{\delta(r)}{4\pi r^2} \, \frac{\overleftrightarrow{\partial}^l}{\partial r^l} \langle \vec{r}|$$

$$g_l = -\frac{4\pi}{m} \left[ \frac{(2l+1)!!}{l!} \right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}} \approx \frac{4\pi}{m} \, \frac{(2l+1)}{(l!)^2} \, a_l^{2l+1}$$

*Application*

- energy-density for inhomogeneous Fermi gases in Thomas-Fermi approximation including p-wave interactions
- exploration of static properties, density profiles, mean-field instability, component separation,...
- collective excitations, vortex dynamics, cooling process,...

# *A Model for Trapped Fermions*

consider a dilute gas of $\Xi$ different fermionic components in an external potential $U(\vec{x})$ at $T = 0\,\mathrm{K}$ interacting via the s- and p-wave part of the Effective Contact Interaction

$$\mathbf{H} = \mathbf{T} + \mathbf{U} + \mathbf{V}_0^{\mathrm{ECI}} + \mathbf{V}_1^{\mathrm{ECI}}$$

$$\mathbf{v}_0^{\mathrm{ECI}} = \frac{4\pi\, a_0}{m} \int \mathrm{d}^3r \, |\vec{r}\rangle \; \frac{\delta(r)}{4\pi r^2} \; \langle\vec{r}|$$

$$\mathbf{v}_1^{\mathrm{ECI}} = \frac{12\pi\, a_1^3}{m} \int \mathrm{d}^3r \, |\vec{r}\rangle \frac{\overleftarrow{\partial}}{\partial r} \frac{\delta(r)}{4\pi r^2} \frac{\overrightarrow{\partial}}{\partial r} \langle\vec{r}|$$

mean-field states &
Thomas-Fermi approximation

energy-density $\mathcal{E}[\kappa_1, \ldots \kappa_\Xi](\vec{x})$ as function of the local Fermi momenta $\kappa_\xi(\vec{x})$ of each component

$$\rho_\xi(\vec{x}) = \frac{1}{6\pi^2}\, \kappa_\xi^3(\vec{x})$$

energy minimization by functional variation of $\kappa_\xi(\vec{x})$ with constrained particle numbers $N_\xi$

extremum condition: coupled polynomial equations for the local Fermi momenta of the groundstate

# *Energy-Density for Trapped Fermions*

## *One-Component System*

$$\mathcal{E}[\kappa](\vec{x}) =$$

$$= \frac{1}{6\pi^2}\, U(\vec{x})\, \kappa^3(\vec{x})$$ — trap —

$$+ \frac{1}{20\pi^2 m}\, \kappa^5(\vec{x})$$ — kinetic —

✗ — s-wave —

$$+ \frac{a_1^3}{30\pi^3 m}\, \kappa^8(\vec{x})$$ — p-wave —

## *Two-Component System*

$$\mathcal{E}[\kappa_1, \kappa_2](\vec{x}) =$$

$$= \frac{1}{6\pi^2}\, [U_1(\vec{x})\, \kappa_1^3(\vec{x}) + U_2(\vec{x})\, \kappa_2^3(\vec{x})]$$

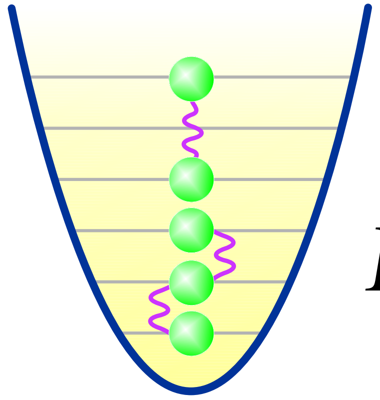
$$+ \frac{1}{20\pi^2 m}\, [\kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x})]$$

$$+ \frac{a_0}{9\pi^3 m}\, \kappa_1^3(\vec{x})\kappa_2^3(\vec{x})$$

$$+ \frac{a_1^3}{30\pi^3 m}\, [\kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) + \\ + \tfrac{1}{2}\kappa_1^3(\vec{x})\kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\kappa_2^3(\vec{x})]$$

- s-wave contact interaction does not contribute due to the Pauli principle
- leading interaction term is p-wave

- s-wave interaction between particles belonging to different components
- p-wave interaction within and between components



*Properties of a*

# *One-Component Fermi Gas*

- Effects of p-Wave Interactions
- Density Profiles
- Mean-Field Instability

# *Effect of p-Wave Interactions*

## *Solution Procedure*

❶ energy-density

$$\mathcal{E}[\kappa](\vec{x}) = \frac{1}{6\pi^2} U(\vec{x})\kappa^3(\vec{x}) + \frac{1}{20\pi^2 m}\kappa^5(\vec{x}) + \frac{a_1^3}{30\pi^3 m}\kappa^8(\vec{x})$$

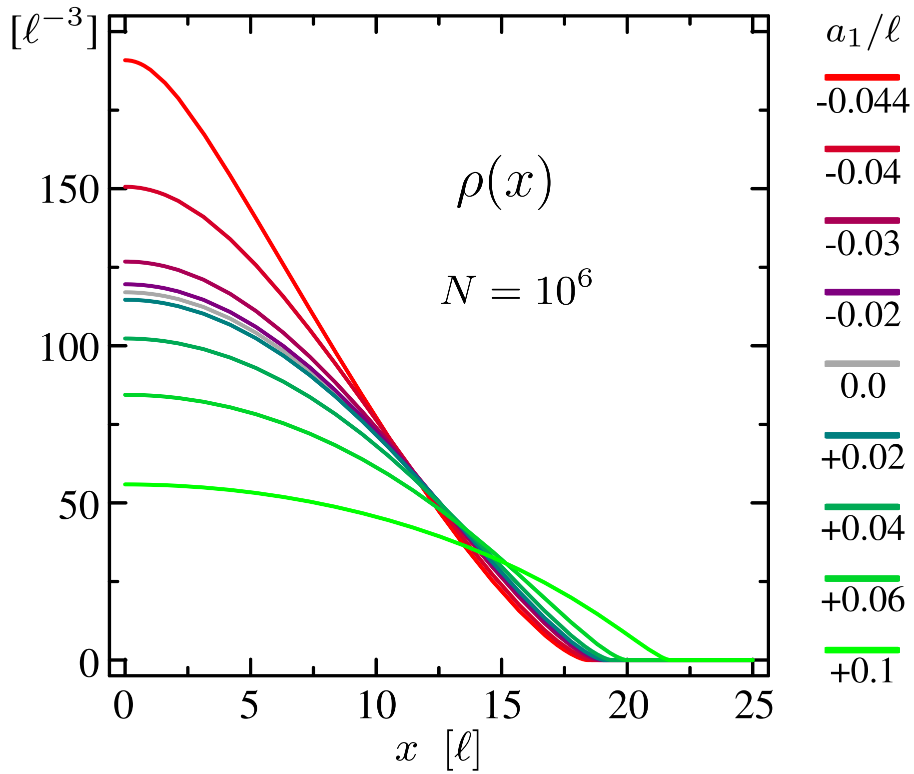
❷ functional variation for fixed particle number leads to extremum condition

$$m[\mu - U(\vec{x})] = \frac{1}{2}\kappa^2(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

❸ choose trapping potential, e.g. harmonic trap with oscillator length $\ell$

$$U(\vec{x}) = \frac{x^2}{2m\ell^4} \qquad \ell = \frac{1}{\sqrt{m\omega}}$$

❹ groundstate density $\rho(\vec{x})$ is obtained by point-wise solution of the extremum condition for given $\mu(N), a_1/\ell$




example: $\ell = 1\mu\text{m}$, $a_1 = 200\, a_{\text{Bohr}}$ $\rightarrow$ $a_1/\ell \approx 0.01$

# p-Wave Attraction & Stability

## Extremum Condition

$$m[\mu - U(\vec{x})] = \frac{1}{2}\kappa^2(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

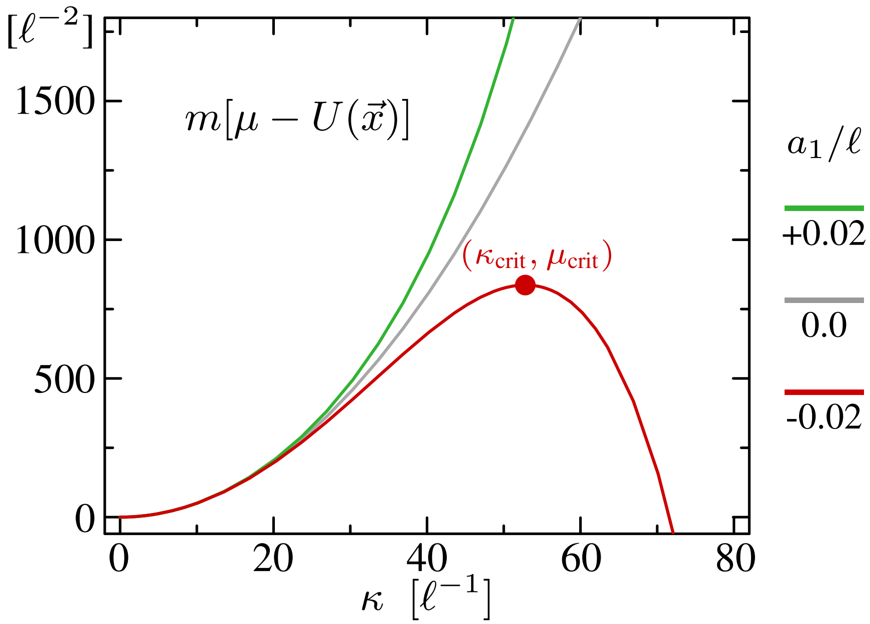
- for attractive p-wave interactions the r.h.s. exhibits a maximum at $(\kappa_{\text{crit}}, \mu_{\text{crit}})$

## Stability Conditions

$$\kappa_{\text{crit}} = \frac{(3\pi)^{1/3}}{2\,|a_1|} \qquad \rho_{\text{crit}} = \frac{1}{16\pi\,|a_1|^3}$$

$$\mu_{\text{crit}} = \frac{3(3\pi)^{3/2}}{40m\,|a_1|^2} \qquad N_{\text{crit}} = \frac{(0.445\,\ell)^6}{|a_1|^6}$$

- there is no stable mean-field solution for densites, chemical potentials or particle numbers beyond the critical values




| | | |
|---|---|---|
| $a_1\ [a_{\text{Bohr}}]$ | -200 | -2000 |
| $a_1/\ell$ | -0.01 | -0.1 |
| $\kappa_{\text{crit}}\ [\mu\text{m}^{-1}]$ | 106 | 10.6 |
| $\rho_{\text{crit}}\ [\mu\text{m}^{-3}]$ | 20000 | 20 |
| $m\mu_{\text{crit}}\ [\mu\text{m}^{-2}]$ | 22000 | 220 |
| $N_{\text{crit}}$ | $7.8 \times 10^9$ | $7.8 \times 10^3$ |

$\ell = 1\mu\text{m}$

# One-Component Fermi Gas
# Variational Picture of Stability

## Variational Ansatz

- trial state is given by the non-interacting Thomas-Fermi solution with classical turning point $\alpha$ as free parameter

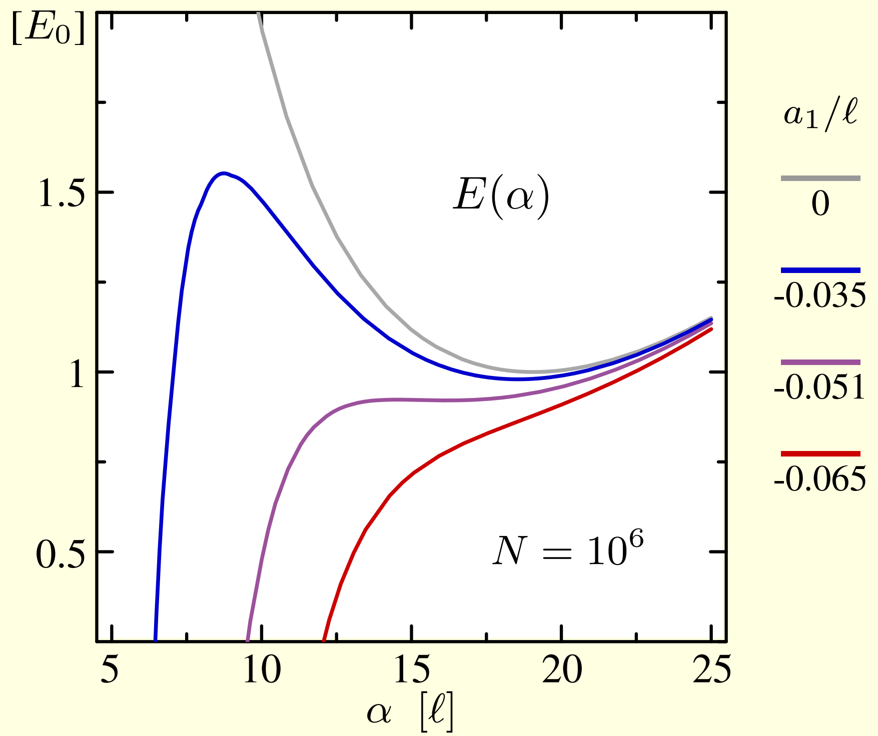
$$\kappa(x) = \frac{2\sqrt[3]{6N}}{\alpha}\sqrt{1-\frac{x^2}{\alpha^2}}$$

- total energy as function of the radius parameter $\alpha$

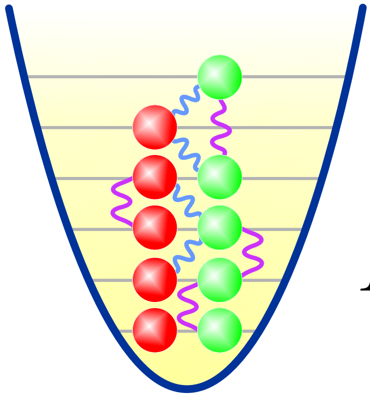
$$E(\alpha; N, a_1, \ell) = C_{\text{trap}} \frac{N\,\alpha^2}{\ell} + C_{\text{kin}} \frac{N^{5/3}}{\alpha^2} + C_{\text{p-wave}} \frac{N^{8/3}\,a_1^3}{\alpha^5}$$

- minimization of the total energy gives (metastable) groundstate of the system

## Mean-Field Collapse




- beyond a critical strength of the attractive interaction the local minimum vanishes
- only the absolute groundstate at very small radii, i.e. very high densities, remains



*Properties of a*

# *Two-Component Fermi Gas*

- Interplay Between s- and p-Wave Interaction
- Mean-Field Instability
- p-Wave Stabilized High-Density Phase
- Component Separation

# *Extremum Condition & Stability*

## *Extremum Condition*

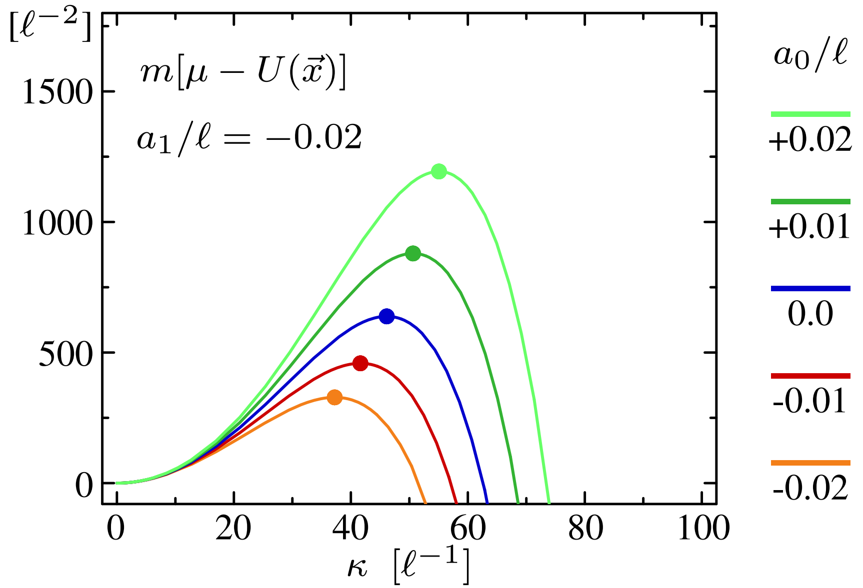
$$m[\mu - U(\vec{x})] = \\ = \frac{1}{2}\kappa^2(\vec{x}) + \frac{2\,a_0}{3\pi}\kappa^3(\vec{x}) + \frac{8\,a_1^3}{15\pi}\kappa^5(\vec{x})$$

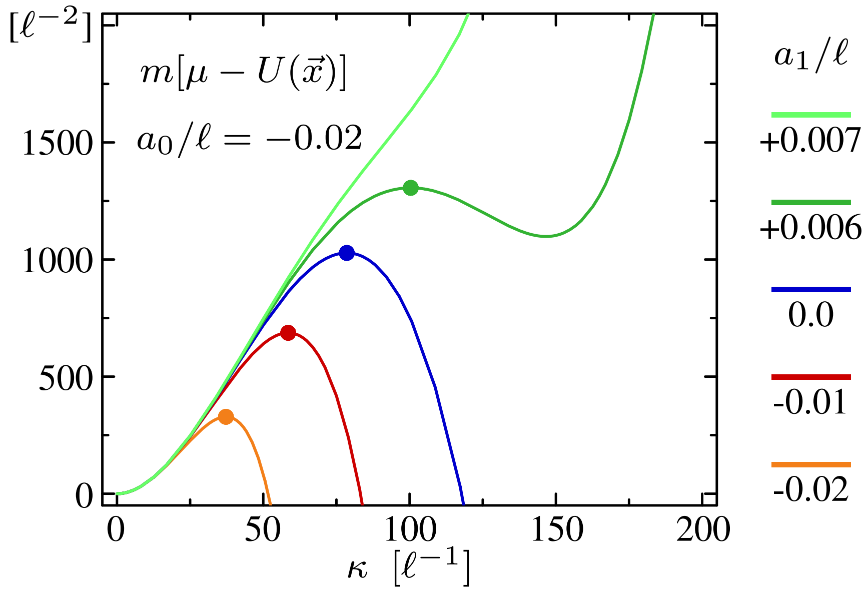
(for simplicity same $[\mu - U(\vec{x})]$ and same $\kappa(\vec{x})$ for both components assumed)

## *Stability Conditions*

$$-a_0\,\kappa_{\text{crit}} - 2(a_1\,\kappa_{\text{crit}})^3 = \frac{\pi}{2}$$

- collapse may occur if one of the scattering lengths is negative
- p-wave repulsion prevents collapse due to s-wave attraction if
$$\frac{a_1}{|a_0|} > \frac{2}{3\pi^{2/3}} \approx 0.31$$
- even a strong s-wave repulsion cannot stabilize the collapse due to p-wave attraction



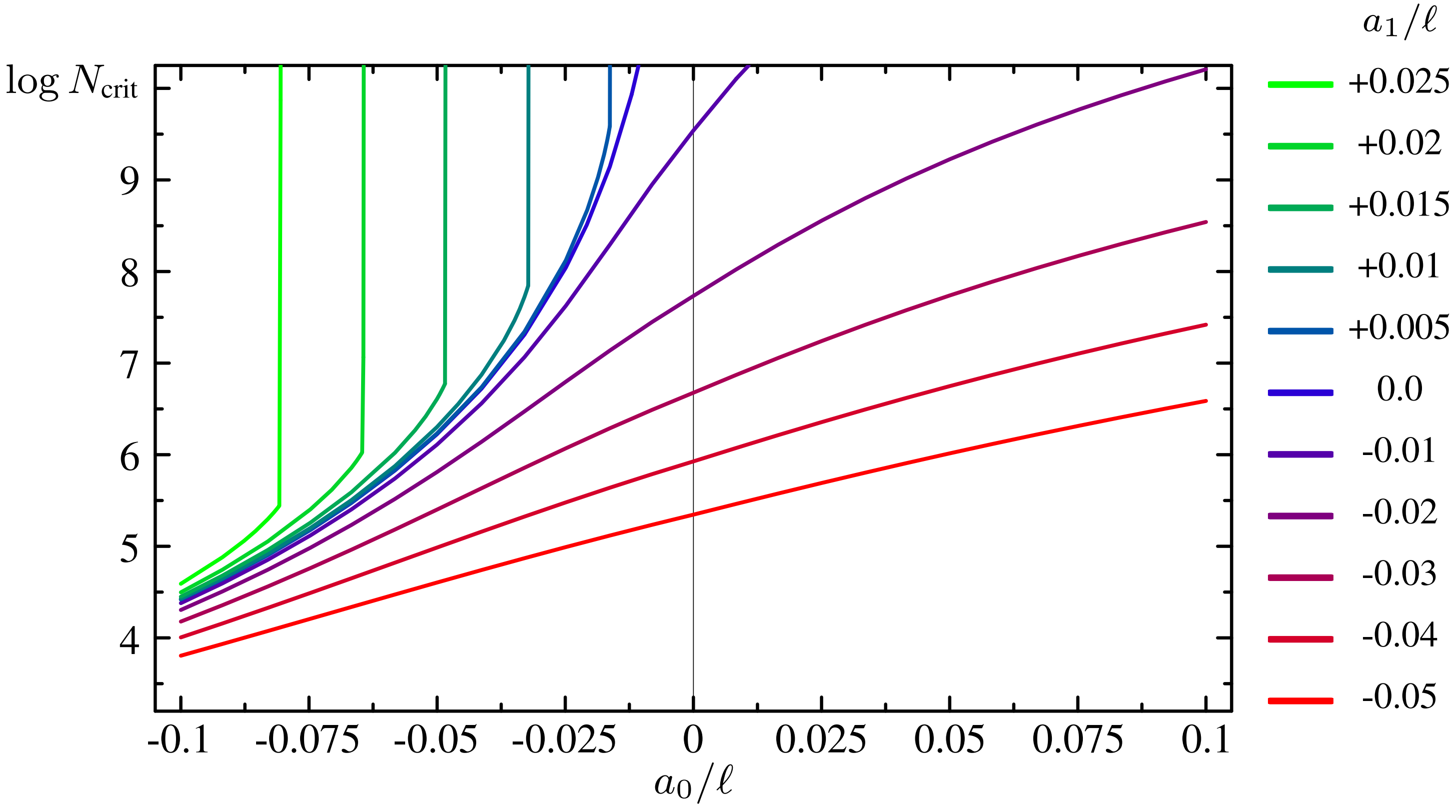
# *Critical Particle Number for Collapse*

- assume harmonic trap with average oscillator length $\ell = \sqrt[3]{\ell_x \ell_y \ell_z}$
- solve extremum condition for the critical chemical potential and calculate $N_{\text{crit}}$

absolute stabilization due to p-wave repulsion $a_1/|a_0| > 2/(3\pi^{2/3})$

p-wave attraction lowers critical particle number substantially

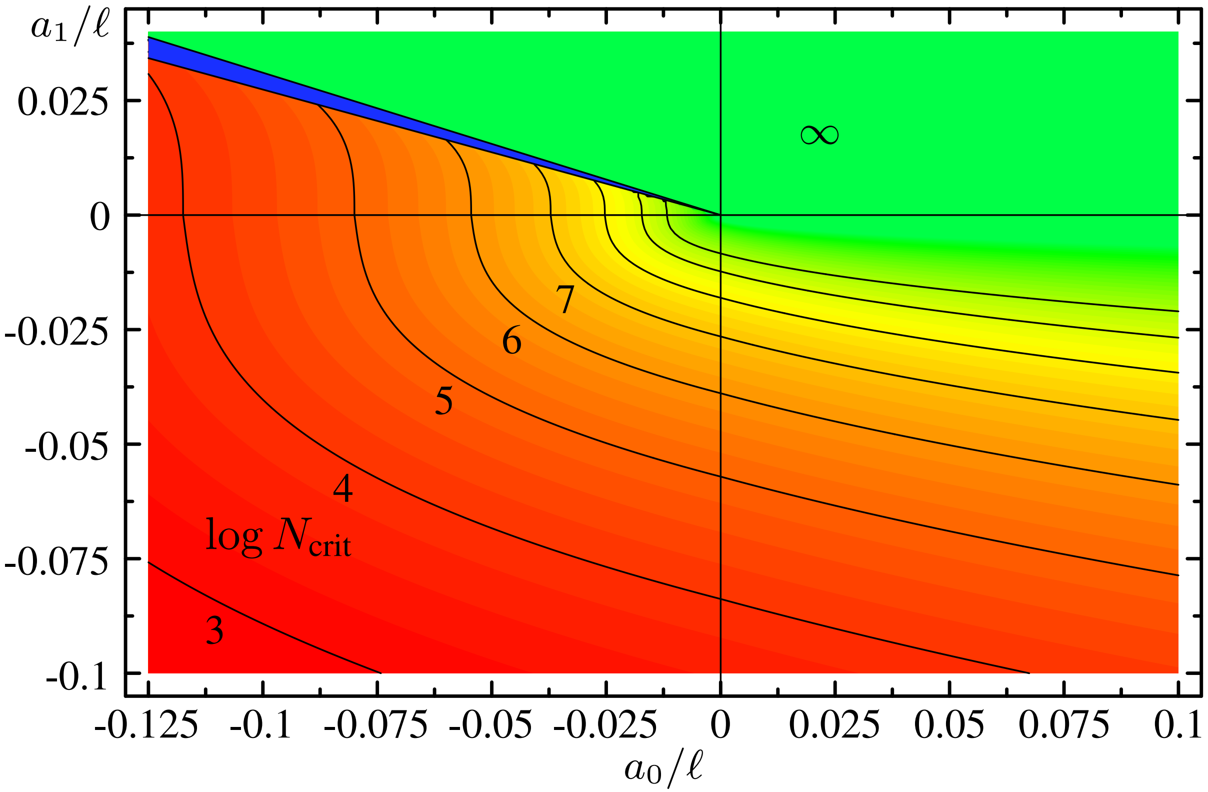
collapse due to p-wave attraction is not stabilized by s-wave repulsion



# Stability Map for Collapse

p-wave stabilized
high-density phase

metastable condensate for
any particle number

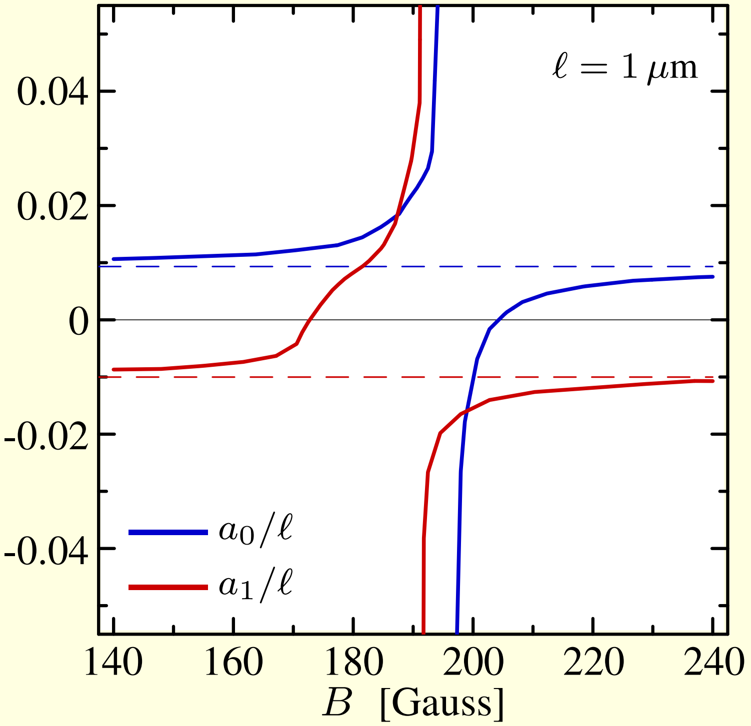



limited particle number for
the metastable condensate

# *Feshbach Resonances in $^{40}$K*

## *s- and p-Wave Feshbach Resonances*

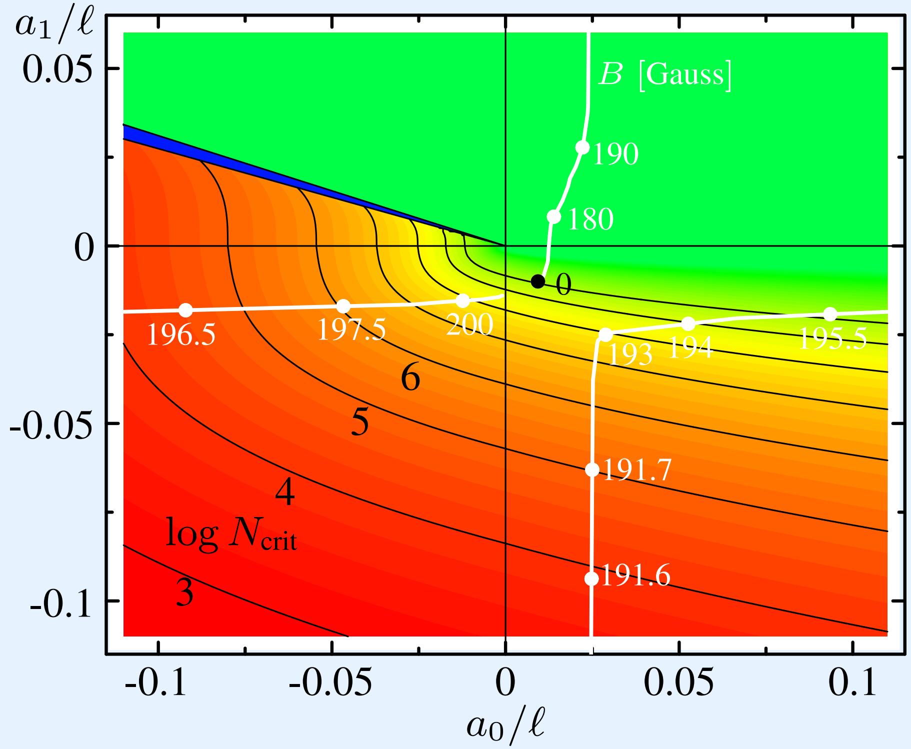
- two-component $^{40}$K system with hyperfine states $F = \frac{9}{2}, m_F = -\frac{9}{2}, -\frac{7}{2}$
- simultaneous s- and p-wave Feshbach resonance theoretically predicted




[J.L. Bohn, Phys. Rev. A61 (2000) 053409]

## *Exploring the Stability Map*

- by changing the strength of the magnetic field one explores nearly the whole stability map
- s- and p-wave interactions have the same importance!



Two-Component Fermi Gas

# p-Wave Stabilized High-Density Phase

## Extremum Condition: $a_0 < 0$, $a_1 > 0$

$$a_1/|a_0| < \frac{2}{3\pi^{2/3}} \approx 0.31$$

- r.h.s. of the extremum condition shows separated low- and high-density branches

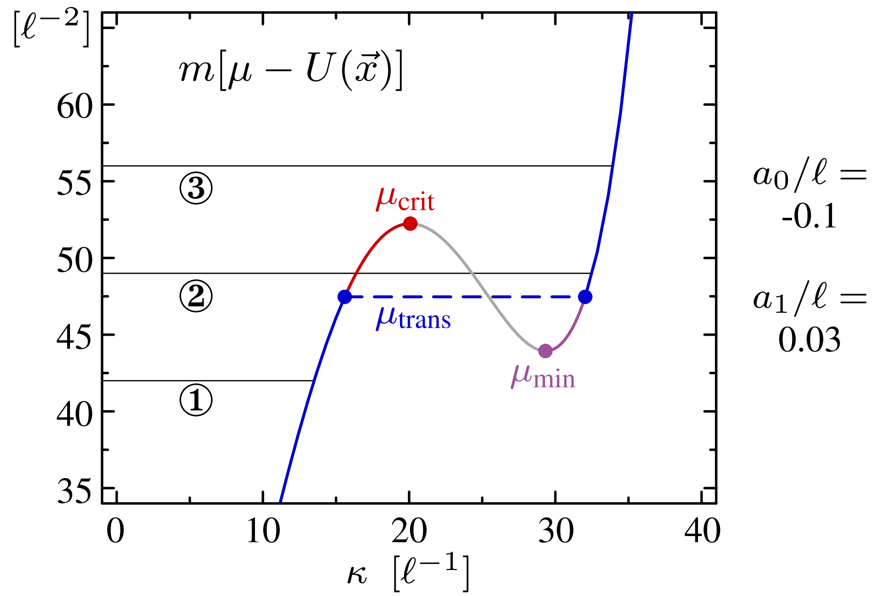
$$a_1/|a_0| > \sqrt[3]{\frac{160}{729\pi^2}} \approx 0.28$$

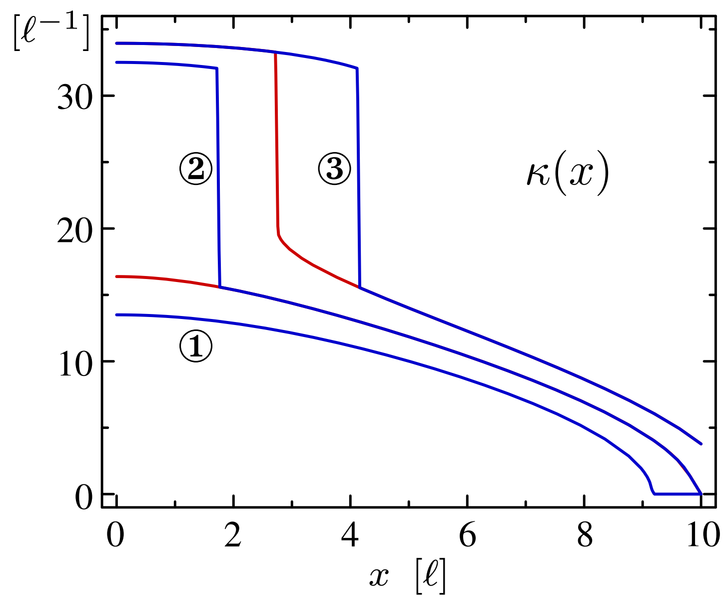
- no self-bound solutions on the high-density branch ($\mu_{\text{min}} > 0$)

## Solution Structure

$[\mu - U(\vec{x})]$

| | |
|---|---|
| $< \mu_{\text{min}}$ | only low-density solution exists |
| $\mu_{\text{min}}$ ... $\mu_{\text{trans}}$ | low- and high-density solution exist; low-density is energetically favored |
| $\mu_{\text{trans}}$ ... $\mu_{\text{crit}}$ | both solutions exist; high-density is energetically favored, low-density is metastable |
| $> \mu_{\text{crit}}$ | only high-density solution exists |

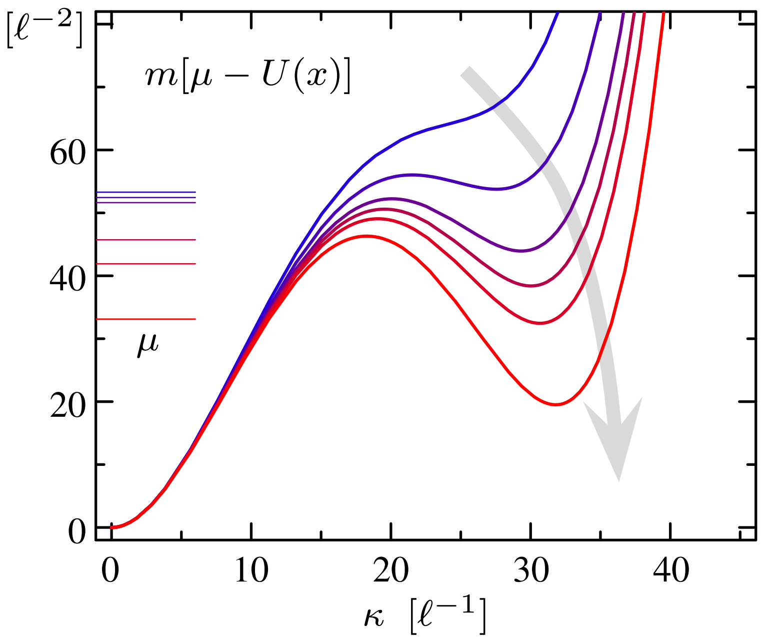


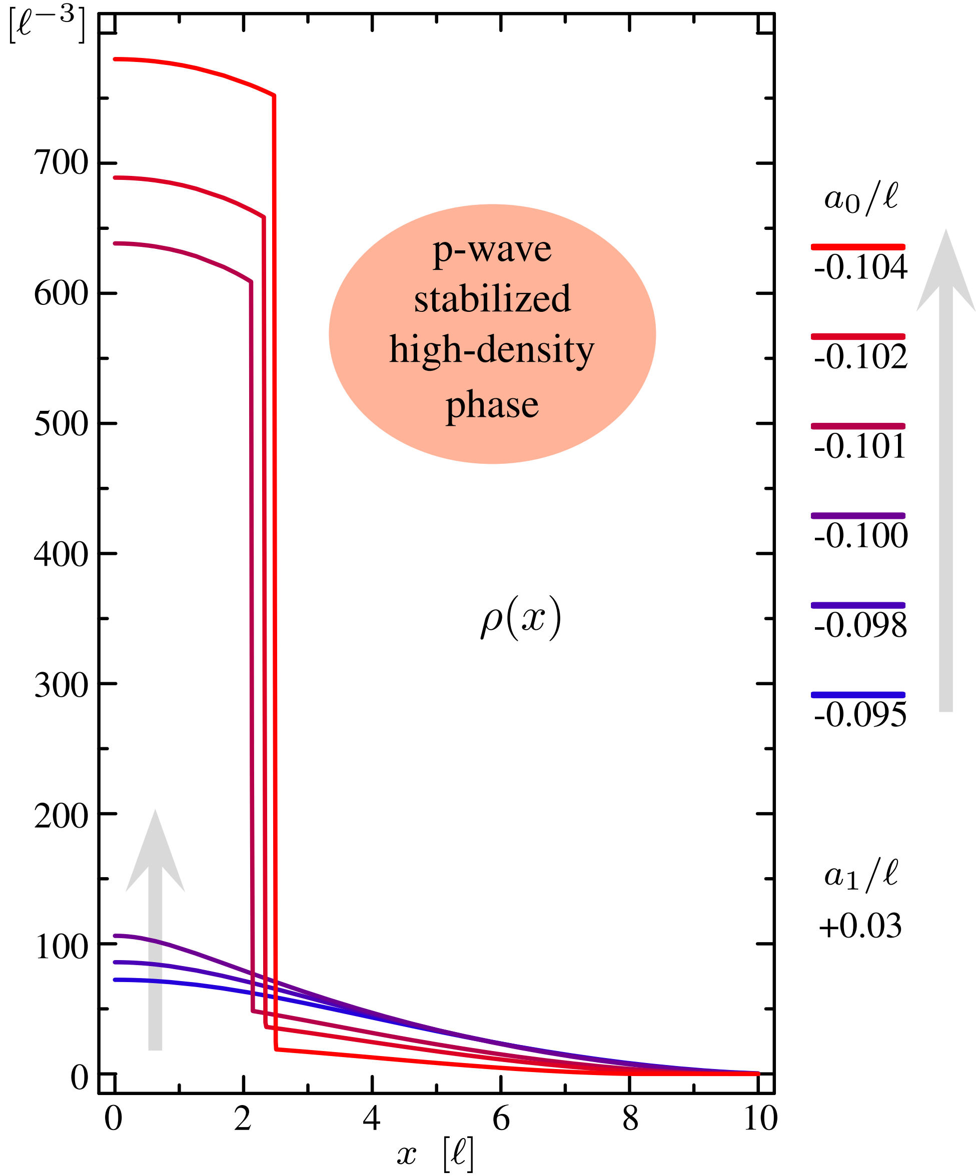



# *High-Density Phase: "Gedankenexperiment"*

## *"Experimental Setup"*

- two-component Fermi gas in a spherical trap with oscillator length $\ell$
- fixed particle number $N_1 = N_2 = 60000$
- ▶ increase the strength of the attractive s-wave interaction adiabatically and keep the repulsive p-wave fixed...



# Component Separation

## Energy-Density

$$\mathcal{E}[\kappa_1,\kappa_2](\vec{x}) = \frac{1}{6\pi^2}[U_1(\vec{x})\,\kappa_1^3(\vec{x}) + U_2(\vec{x})\,\kappa_2^3(\vec{x})] + \frac{1}{20\pi^2 m}[\kappa_1^5(\vec{x})+\kappa_2^5(\vec{x})] + \frac{a_0}{9\pi^3 m}\kappa_1^3(\vec{x})\kappa_2^3(\vec{x}) + \frac{a_1^3}{30\pi^3 m}[\kappa_1^8(\vec{x})+\kappa_2^8(\vec{x}) + \tfrac{1}{2}\kappa_1^3(\vec{x})\kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\kappa_2^3(\vec{x})]$$

- for repulsive s-wave (p-wave) interactions it may be energetically favorable to separate both components spatially

## Overlapping Configuration

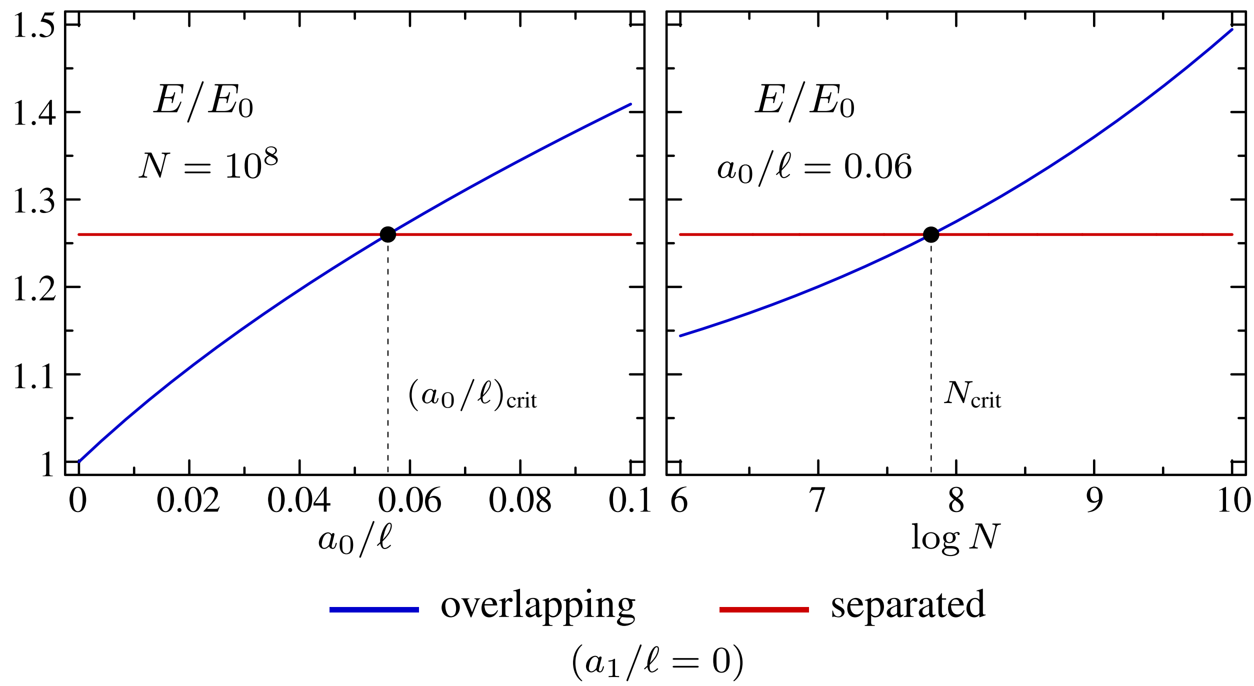
$$\kappa_1(\vec{x}) \equiv \kappa_2(\vec{x})$$

- ► s- and p-wave interaction terms contribute

## Separated Configuration

$$\kappa_1(\vec{x})\kappa_2(\vec{x}) \equiv 0$$

- ► s-wave interaction does not contribute at all



*Two-Component Fermi Gas*

# *Critical Particle Number for Separation*

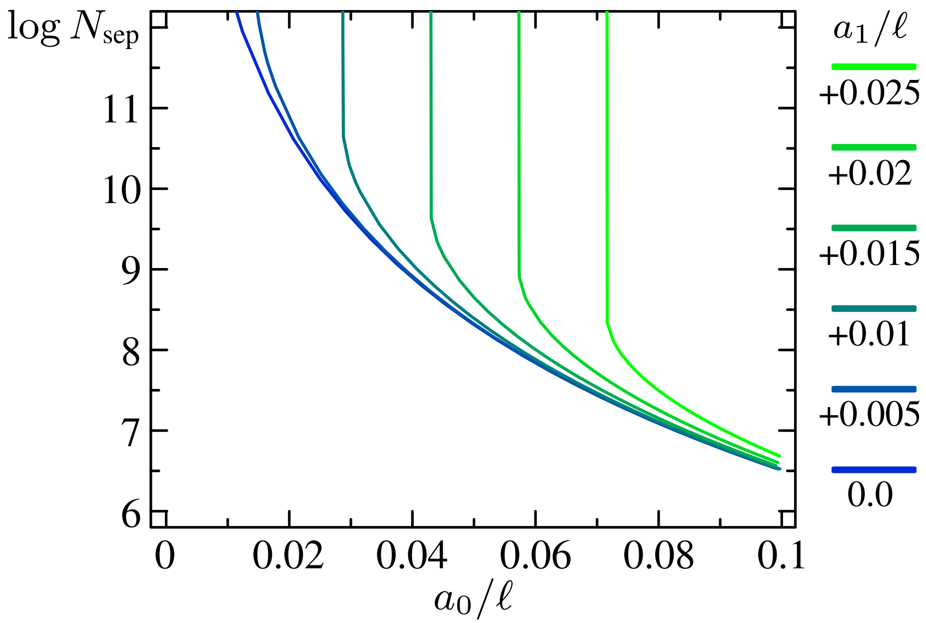
- assume harmonic trap with average oscillator length $\ell = \sqrt[3]{\ell_x \ell_y \ell_z}$
- choose $\mu$ such that the particle numbers and the energy of the overlapping and the separated configuration are equal

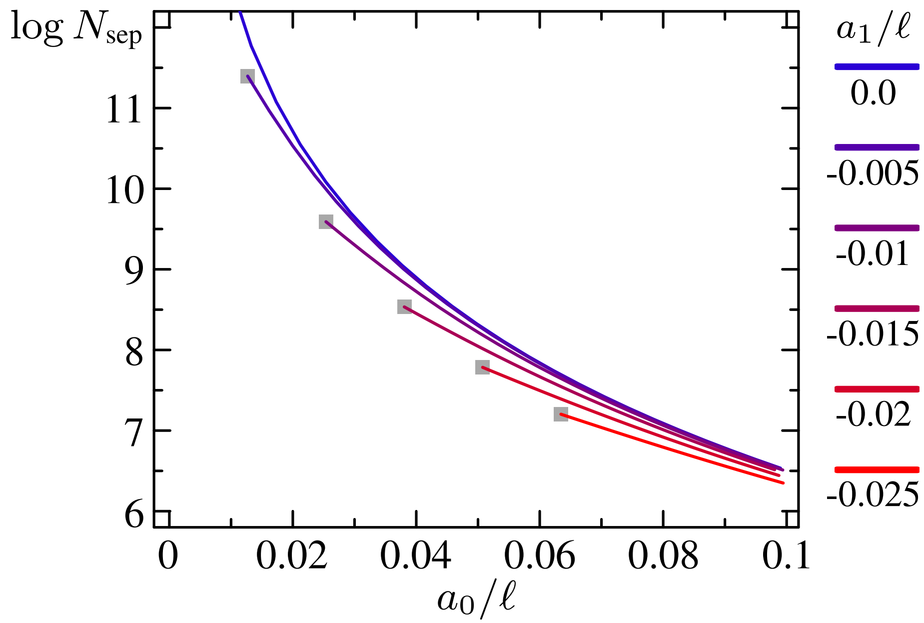
p-wave repulsion raises critical particle number

absolute stabilization due to p-wave repulsion $a_1/a_0 > 0.349$

p-wave attraction lowers critical particle number

collapse of the separated phase due to p-wave attraction

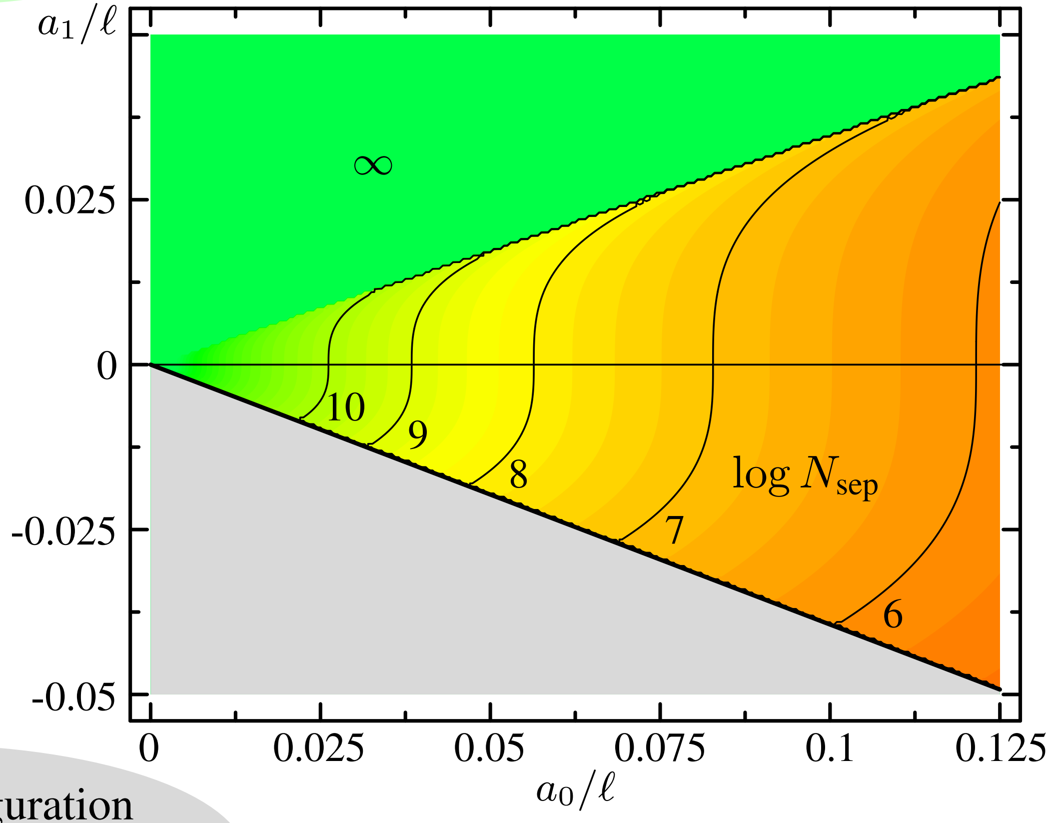


*Two-Component Fermi Gas*

# *Stability Map for Component Separation*

overlapping configuration is lower in energy for any particle number

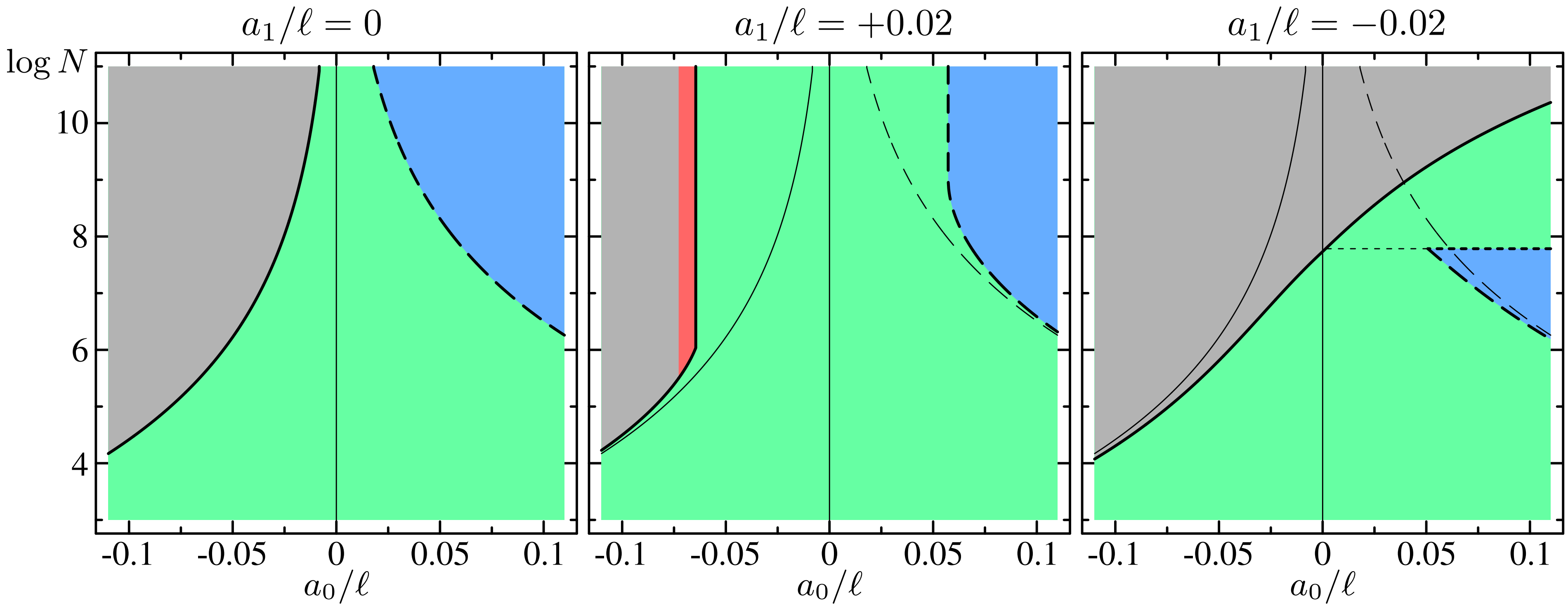



beyond $N_{sep}$ the separated configuration is energetically favored

separated configuration instable due to p-wave attraction

## *Two-Component Fermi Gas*

# *“Phase Diagrams”*




metastable state with identical overlapping density profiles

metastable state with spatially separated components

unstable against collapse to a high-density state

p-wave stabilized high-density phase in the center of the trap

# *Summary*

*Strategy*

- derived an Effective Contact Interaction that reproduces the two-body spectrum
- used s- and p-wave terms to set up the energy-density of an inhomogeneous Fermi gas in Thomas-Fermi approximation
- investigated the effects of s- and p-wave interactions on the structure and stability of one- and two-component systems

*Results*

- attractive p-wave interactions limit the particle number/ density of the one-component system (s-wave interactions do not contribute)
- complex interplay between s- and p-wave interactions with respect to collapse and component separation in the two-component system
- novel phenomena: absolute stabilization due to p-wave repulsion, p-wave stabilized high-density phase

► **do not neglect p-wave interactions from the outset!**

...have a look at
http://theory.gsi.de/~trap