Structure of Ultracold Trapped Fermi and Bose Gases

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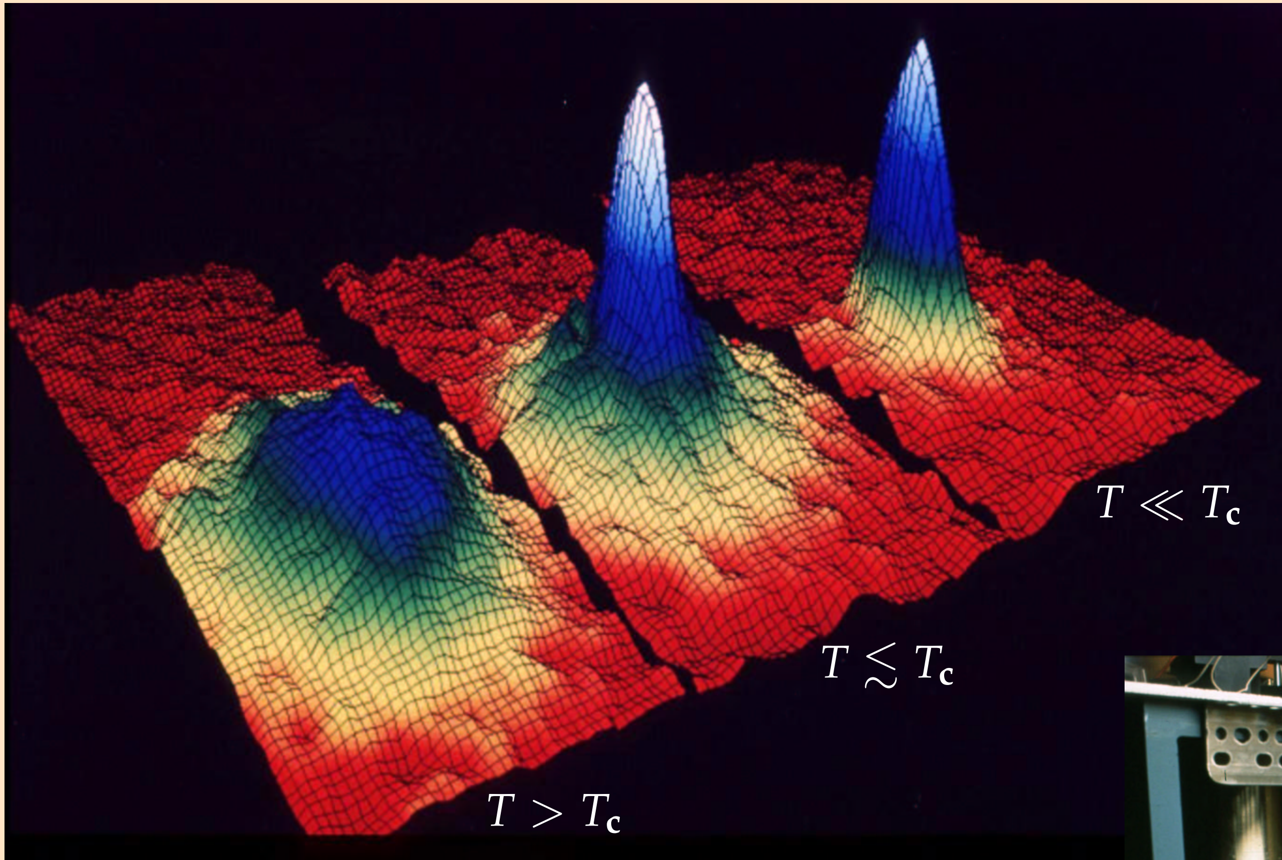
Overview

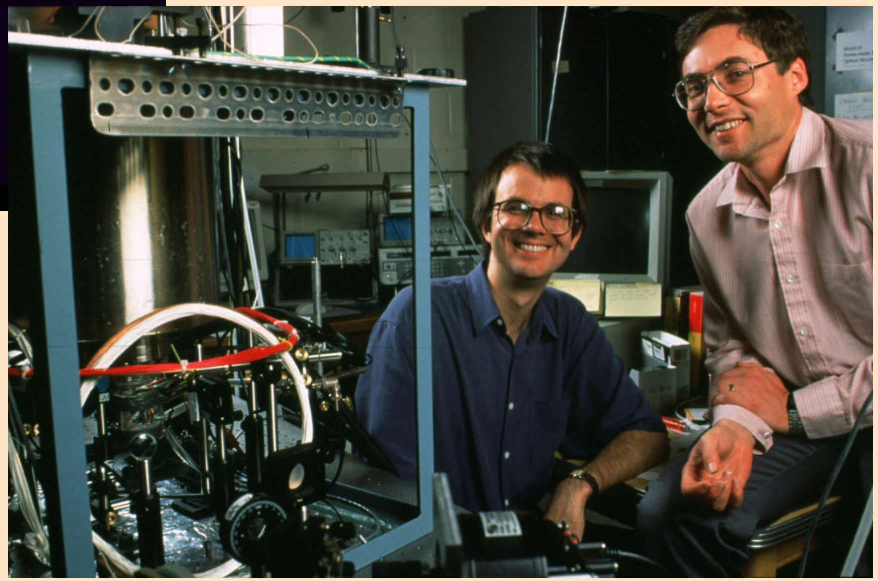
- The World of Trapped Ultracold Atomic Gases
- Theoretical Description of Trapped Ultracold Gases
  - The Many-Body Problem
  - Correlations & Effective Interaction
  - Mean-Field & Thomas-Fermi Approximation
  - Energy Functional
- Structure of a Trapped Degenerate Fermi Gas
  - Energy Landscapes & Density Profiles
  - Mean-Field Induced Collapse
  - Component Separation
  - Phase Diagram

Boulder / Colorado — June 5th, 1995 — 10:54 am

BEC of Rubidium Atoms



- ^{87}Rb ($F=2, m_F=2$)
- $N_{\text{initial}} \approx 10^6$
- $N_{\text{BEC}} \approx 2000$
- $T_c \approx 170$nK
- absorption image after 60 ms expansion
- 0.2mm × 0.27mm

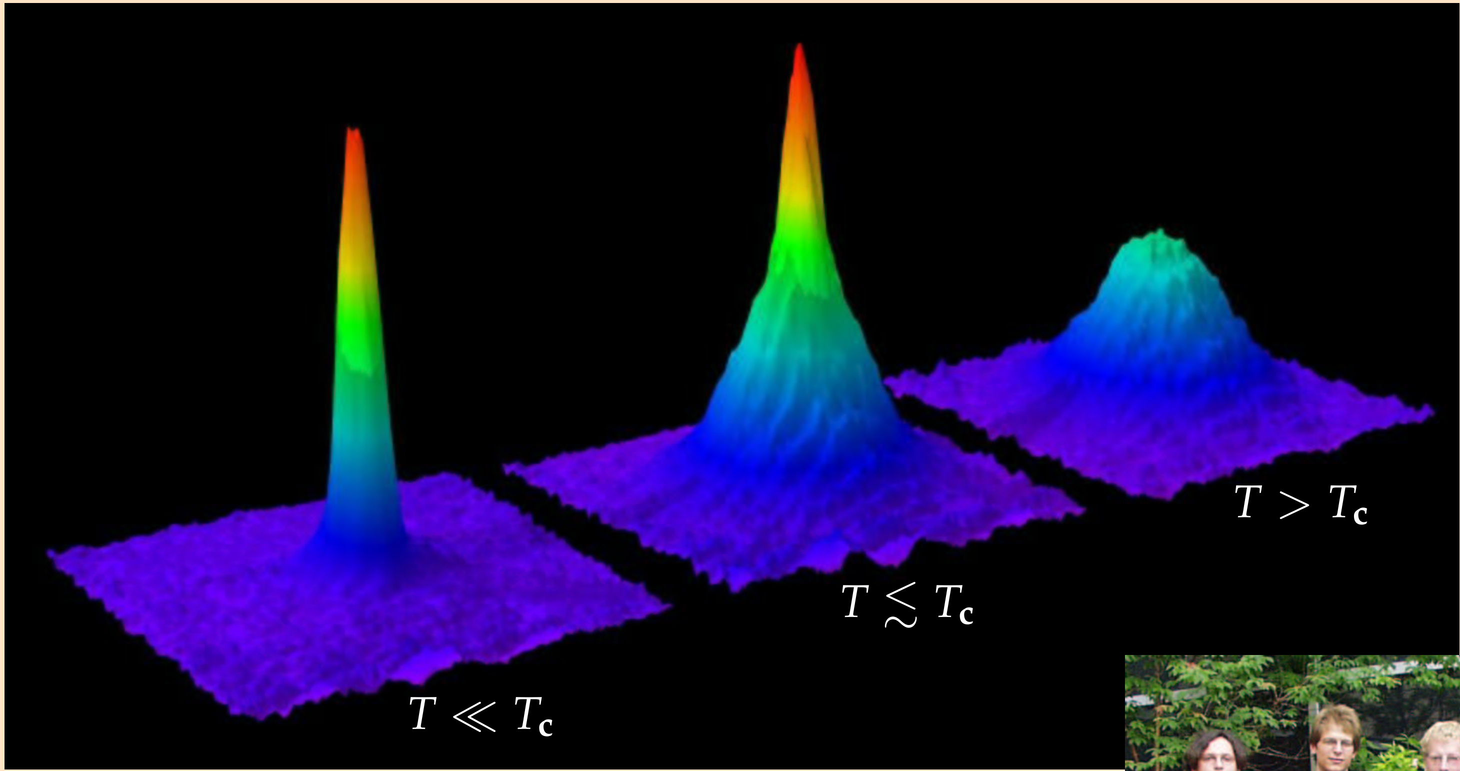


E. Cornell, C. Wieman, et al.
(JILA, NIST, U of Colorado)

Nobel Prize in Physics 2001

Cambridge / Massachusetts — September 1995

BEC with Sodium Atoms



- ^{23}Na $(F=1, m_F=-1)$
- $N_{\text{initial}} \approx 10^9$
- $N_{\text{BEC}} \approx 5 \times 10^5$
- $T_c \approx 2\,\mu$K
- absorption image after 60 ms expansion
- 1mm × 1mm

W. Ketterle, et al.
(MIT)

Nobel Prize in Physics 2001

Why are they so Attractive...

a macroscopic system which exhibits quantum properties

all relevant quantities are **observable** & **tunable**

size, density, particle number, mass, statistic, composition, temperature, distributions, interaction strength...

large composite bosons/fermions

meta stable many-body state

realization of a dilute quantum gas

interatomic interactions are “weak”

BE condensation
$T_{\mathrm{BEC}} \sim \mu\mathrm{K}$

BCS transition
$T_{\mathrm{BCS}} \sim \mathrm{nK}$

Experimentalists' Toolbox

Trapping

- **magneto-optical trap**: absorption of counterpropagating photons and reemission leads to net restoring force
- **magnetic trap**: inhomogeneous magnetic field couples to the magnetic moment of the atoms; trapped in field minimum
- **optical traps**

Cooling

- **Doppler cooling**: atom absorbs a red detuned photon and reemits an on-resonance photon; energy difference is paid with kinetic energy
- **evaporative cooling**: selective removal of high-energy atoms and thermalization results in lower temperature of the remanent

Imaging

- **resonance absorption**: shine resonant laser onto the cloud and observe shadow
- **phase-contrast**: use off-resonant laser and observe interference of scattered and unscattered light
- **ballistic expansion**: momentum distribution can be determined

Manipulation

- **Feshbach resonances** allow tuning of the strength of inter-atomic interactions (scattering length) over a wide range
- **optical tools**: e.g. focused blue detuned lasers 'repel' the atoms and can be used to cut or stir the gas

Trapped Degenerate Fermi Gas

Science 285 (1999) 1703

Onset of Fermi Degeneracy in a Trapped Atomic Gas

B. DeMarco and D. S. Jin*†

An evaporative cooling strategy that uses a two-component Fermi gas was employed to cool a magnetically trapped gas of 7 × 10^5 ^{40}K atoms to 0.5 of the Fermi temperature T_F. In this temperature regime, where the state occu- ... the lowest energies has increased from essentially zero ... nearly 60 percent, quantum degeneracy was ... tive cooling and as a modification of the t... the mo... the tota... gas di... ...cs.

cooling a cloud of neutral ^{40}K atoms kept in a magnetic trap

^{40}K has fractional total spin: **fermion**

$$F = \underset{I}{4} \pm \underset{S}{1/2} = \tfrac{9}{2}, \tfrac{7}{2}$$

two-component mixture

$$|F = \tfrac{9}{2}, m_F = \tfrac{9}{2}\rangle$$
$$|F = \tfrac{9}{2}, m_F = \tfrac{7}{2}\rangle$$

$N \approx 10^5 \ldots 10^6$

$\rho \approx 10\,\mu\mathrm{m}^{-3}$

$\ell \approx 1\,\mu\mathrm{m}$

$\tau \approx 300\,\mathrm{s}$

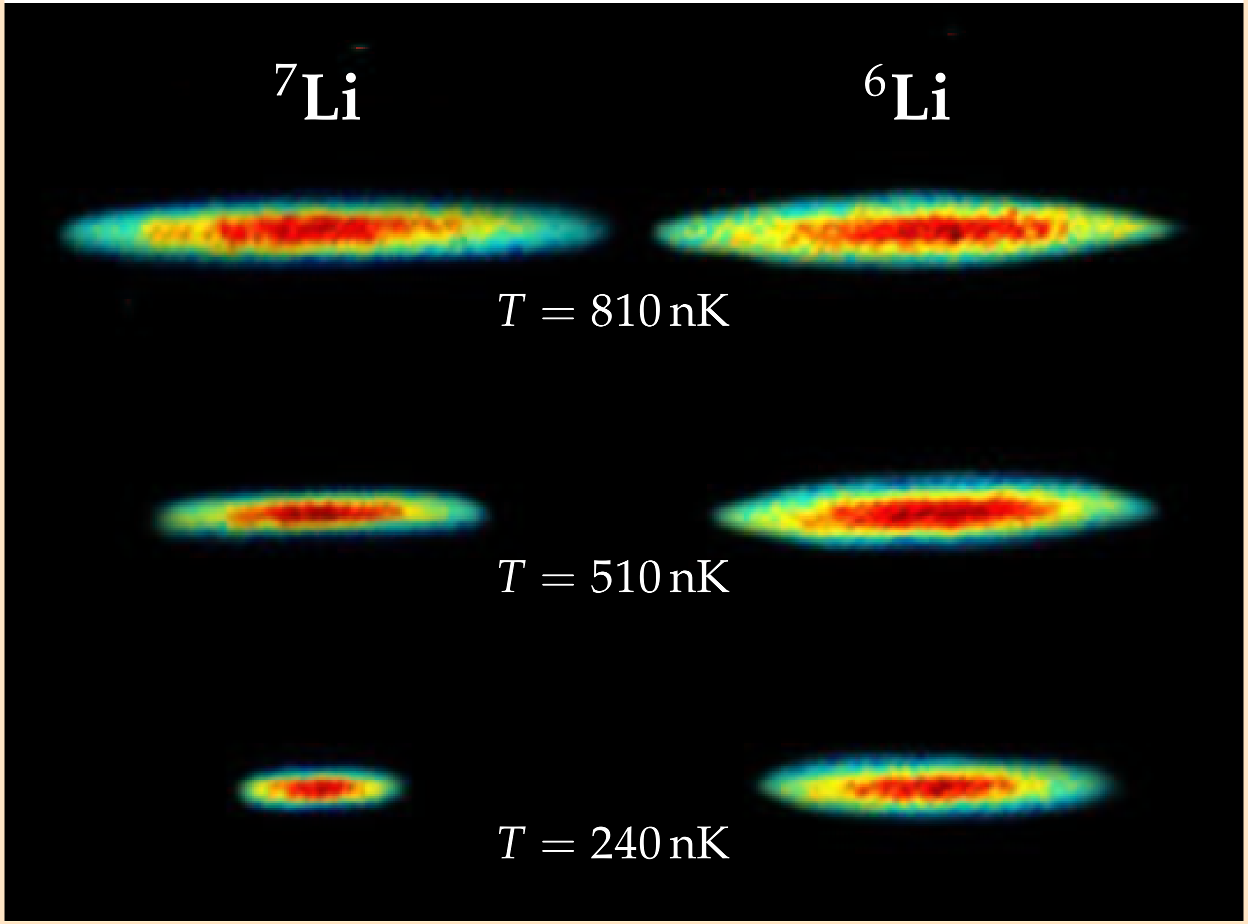
$T \approx 300\,\mathrm{nK}$
$\approx 0.5\,\varepsilon_F$

Degenerate Boson-Fermion Mixtures

Science 291 (2001) 2570

Observation of Fermi Pressure in a Gas of Trapped Atoms

Andrew G. Truscott, Kevin E. Strecker, William I. McAlexander,*
Guthrie B. Partridge, Randall G. Hulet†



simultaneous trapping of
$^7\mathrm{Li}\ (F = 2) \Rightarrow$ **boson**
$^6\mathrm{Li}\ (F = \frac{3}{2}) \Rightarrow$ **fermion**

evaporative cooling of the bosons $\longrightarrow$ sympathetic cooling of the fermions

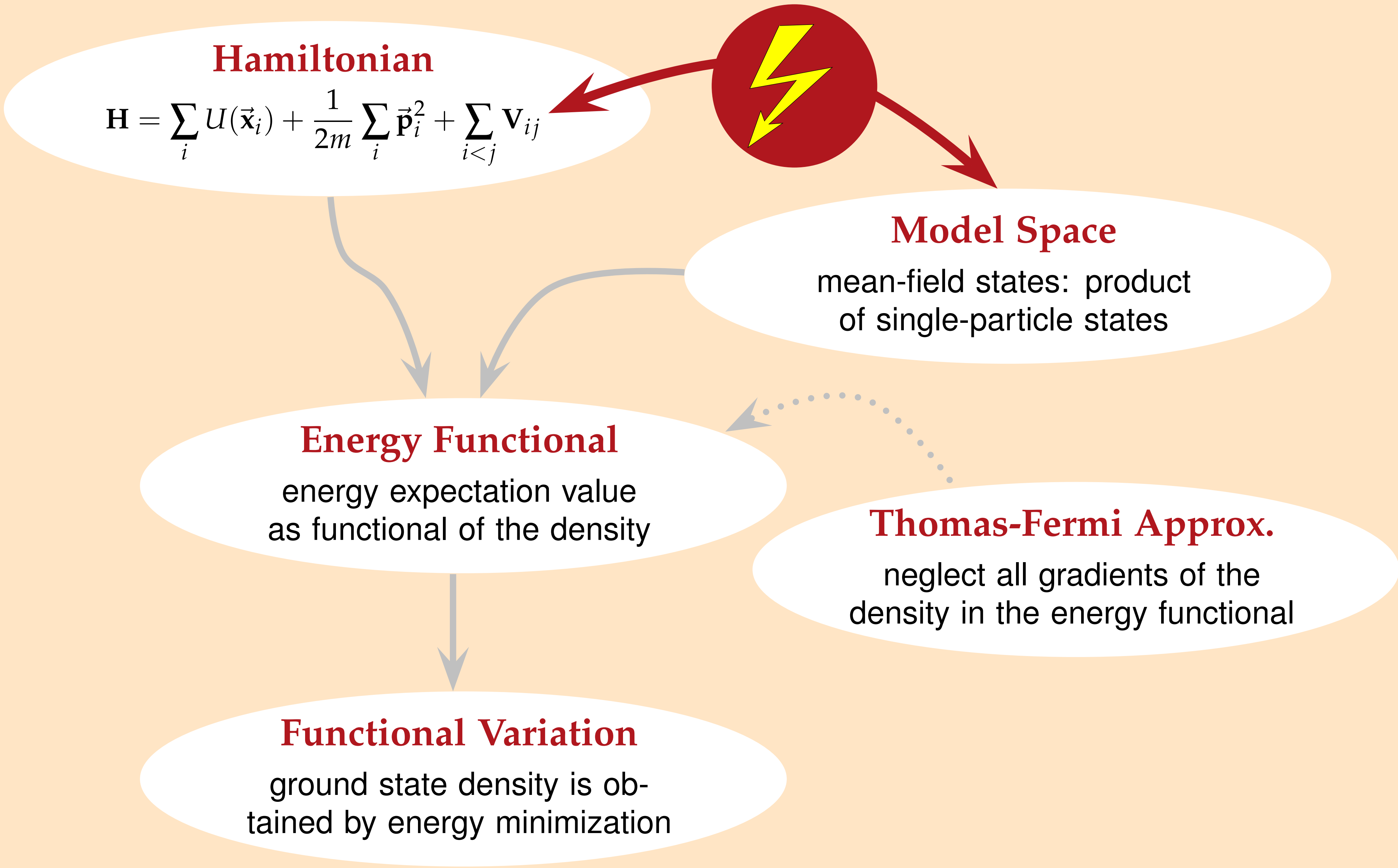
$N_\mathrm{B} \approx N_\mathrm{F} \approx 10^5 \ldots 10^6$

$T \approx 240\,\mathrm{nK}$
$\approx 0.25\,\varepsilon_F$

Theoretical Description of Trapped Ultracold (Fermi) Gases

- **The Many-Body Problem**
- **Correlations & Effective Interaction**
- **Mean-Field & Thomas-Fermi Approximation**
- **Energy Functional**

Route Through the Many-Body Problem



Short-Range Correlations

Interaction

many realistic two-body interactions show a strong short-range repulsion

(e.g. nucleon-nucleon & van der Waals interactions)

Correlations

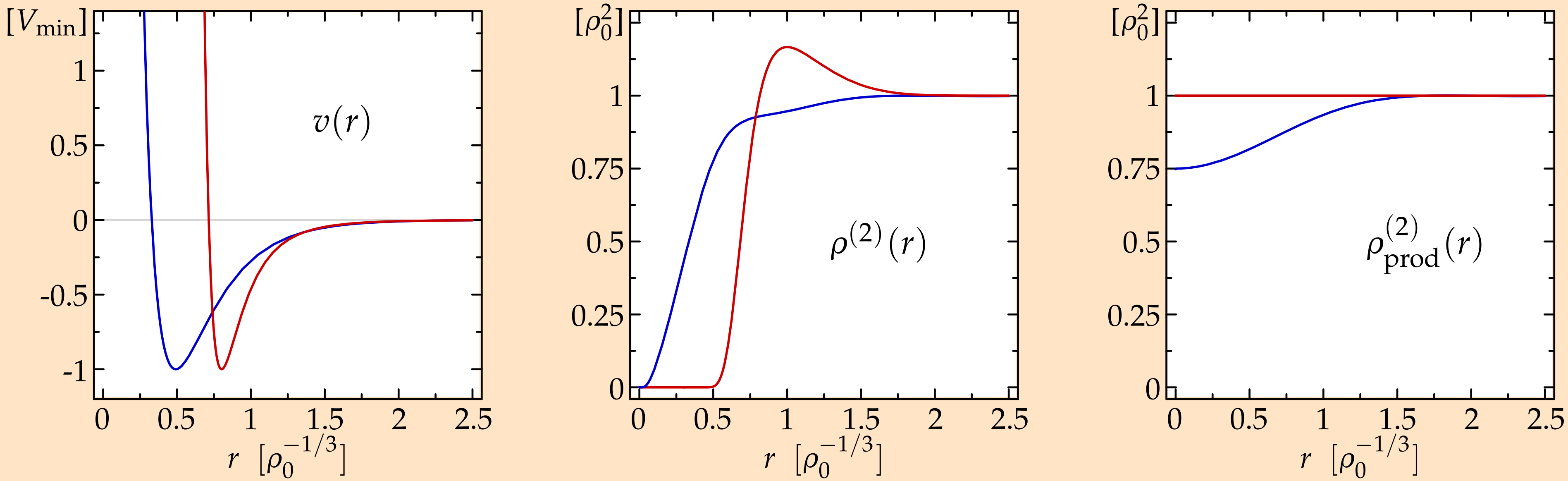
core induces strong short-range correlations in many-body state

(e.g. correlation hole in two-body density)

Product States

short-range correlations cannot be described by product-type states

(e.g. mean-field, superposition of few product states,...)



nuclear matter $\rho_0 = 0.17\,\text{fm}^{-3}$

liquid ^{4}He (bosonic) $\rho_0 = 0.022\,\text{Å}^{-3}$

Short-Range Correlations

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short-range correlations cannot be described by product-type states

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Effective Interaction

replace the full potential by a tamed effective interaction

Correlated States

include correlations in many-body model-space

Effective Contact Interaction

A Suitable Effective Interaction...

system is very **dilute** and **cold**

$\rho^{-1/3} \gg$ range of interaction

$q^{-1} \gg$ range of interaction

treat the many-body problem in a restricted **model-space** that does not contain correlations

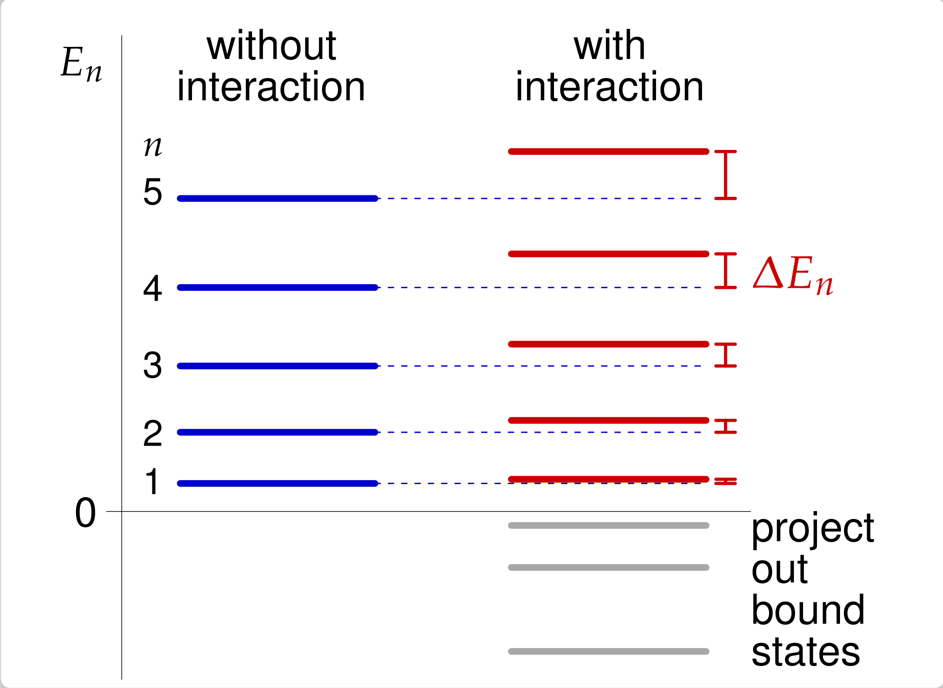
looking for the structure of **non-selfbound states** in an external potential

hermitean interaction operator that obeys standard symmetries (translation, rotation,...)

Effective Contact Interaction (ECI)

- zero-range potential (for each partial wave)
- expectation value in two-body model-states equals the energy shift induced by the full interaction

$$\left\langle \phi_n^{\text{mod}} \right| \mathbf{v}^{\text{ECI}} \left| \phi_n^{\text{mod}} \right\rangle \stackrel{!}{=} \Delta E_n$$



Construction of the ECI

Energy Shift

- relative two-body wave function w/o and with interaction for $r >$ range of $v(r)$

$$\phi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\vartheta, \varphi)$$

$$R_{nl}(r) \propto j_l(q_{nl} r)$$
$$\bar{R}_{nl}(r) \propto j_l(\bar{q}_{nl} r) - \tan \eta_l(\bar{q}_{nl})\, n_l(\bar{q}_{nl} r)$$

- auxiliary boundary condition $R_{nl}(\Lambda) = 0$ to obtain discrete momentum spectrum

$$q_{nl}\Lambda = \pi(n + \tfrac{l}{2})$$
$$\bar{q}_{nl}\Lambda = \pi(n + \tfrac{l}{2}) - [\eta_l(\bar{q}_{nl}) - \pi\, n_l^{\text{bound}}]$$

- momentum shift

$$\Delta q_{nl}\Lambda = (\bar{q}_{nl} - q_{nl})\Lambda$$
$$= -[\eta_l(q_{nl}) - \pi\, n_l^{\text{bound}}] =: -\hat{\eta}_l(q_{nl})$$

- relative energy shift

$$\frac{\Delta E_{nl}}{E_{nl}} = -\frac{2}{q_{nl}\Lambda}\, \hat{\eta}_l(q_{nl})$$

Interaction Operator

- ansatz for a nonlocal contact interaction for the lth partial wave

$$\mathbf{v}_l^{\text{ECI}} = (\vec{\mathbf{q}} \cdot \tfrac{\vec{\mathbf{r}}}{\mathbf{r}})^l\, g_l\, \delta^{(3)}(\vec{\mathbf{r}})\, (\tfrac{\vec{\mathbf{r}}}{\mathbf{r}} \cdot \vec{\mathbf{q}})^l$$

$$= \int d^3r\, |\vec{r}\rangle \frac{\overleftrightarrow{\partial}^l}{\partial r^l}\, g_l\, \delta^{(3)}(\vec{r})\, \frac{\overrightarrow{\partial}^l}{\partial r^l} \langle \vec{r}|$$

- expectation value in non-interacting two-body states

$$\langle \phi_{nlm} |\, \mathbf{v}_l^{\text{ECI}}\, | \phi_{nlm} \rangle \stackrel{!}{=} \Delta E_{nl}$$

- interaction strengths g_l determined by $\hat{\eta}_l(q)$

$$g_l = -\frac{4\pi}{2\mu} \left[\frac{(2l+1)!!}{l!}\right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}}$$

- parametrization of $\hat{\eta}_l(q)$ in terms of the scattering lengths a_l for $|q\, a_l| \ll 1$

$$g_l = \frac{4\pi}{2\mu} \frac{(2l+1)}{(l!)^2}\, a_l^{2l+1} + \mathcal{O}(q^2)$$

A Model for a

Trapped Degenerate Fermi Gas

- trapped gas of Ξ distinguishable fermionic species ($\xi = 1, ..., \Xi$) interacting via the s- and p-wave contact interaction
- for simplicity: equal trapping potentials and s- and p-wave scattering lengths, a_0 and a_1, for all components

Hamiltonian

$$\mathbf{H} = \underbrace{\sum_i U(\vec{\mathbf{x}}_i)}_{\text{trap}} + \underbrace{\frac{1}{2m}\sum_i \vec{\mathbf{p}}_i^2}_{\text{kinetic}} + \underbrace{\frac{4\pi a_0}{m}\sum_{i<j}\delta^{(3)}(\vec{\mathbf{r}}_{ij})}_{\text{s-wave}} + \underbrace{\frac{12\pi a_1^3}{m}\sum_{i<j}\left(\vec{\mathbf{q}}_{ij}\cdot\frac{\vec{\mathbf{r}}_{ij}}{\mathbf{r}_{ij}}\right)\delta^{(3)}(\vec{\mathbf{r}}_{ij})\left(\frac{\vec{\mathbf{r}}_{ij}}{\mathbf{r}_{ij}}\cdot\vec{\mathbf{q}}_{ij}\right)}_{\text{p-wave}}$$

Mean-Field States (homogeneous)

- N-body state $|\Psi\rangle$ is an antisymmetrized product of single-particle momentum eigenstates $|\vec{k}_i, \xi_i\rangle$

$$|\Psi\rangle = \mathcal{A}\left(|\vec{k}_1, \xi_1\rangle \otimes \cdots \otimes |\vec{k}_N, \xi_N\rangle\right)$$

- for each component ξ all momenta $|\vec{k}|$ up to the **Fermi momentum** κ_ξ appear

Thomas-Fermi Approximation

- energy density of the trapped gas is locally given by the energy density of the homogeneous system

$$\mathcal{E}_{\text{hom}}(\kappa_1, ..., \kappa_\Xi) = \frac{1}{V}\langle\Psi|\,\mathbf{H}_{\text{hom}}\,|\Psi\rangle$$

- i.e. the Fermi momenta κ_ξ are replaced by **local Fermi momenta** $\kappa_\xi(\vec{x})$

Energy-Density for Trapped Fermions

Single-Component System

$$\mathcal{E}_1[\kappa](\vec{x}) =$$
$$= \frac{1}{6\pi^2}\, U(\vec{x})\, \kappa^3(\vec{x})$$
$$+ \frac{1}{20\pi^2 m}\, \kappa^5(\vec{x})$$
✗
$$+ \frac{a_1^3}{30\pi^3 m}\, \kappa^8(\vec{x})$$

— trap —

— kinetic —

— s-wave —

— p-wave —

Two-Component System

$$\mathcal{E}_2[\kappa_1, \kappa_2](\vec{x}) =$$
$$= \frac{1}{6\pi^2} U(\vec{x})\, \left[\kappa_1^3(\vec{x}) + \kappa_2^3(\vec{x})\right]$$
$$+ \frac{1}{20\pi^2 m}\, \left[\kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x})\right]$$
$$+ \frac{a_0}{9\pi^3 m}\, \kappa_1^3(\vec{x})\, \kappa_2^3(\vec{x})$$
$$+ \frac{a_1^3}{30\pi^3 m}\, \Big[\kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) +$$
$$+ \tfrac{1}{2}\kappa_1^3(\vec{x})\, \kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\, \kappa_2^3(\vec{x})\Big]$$

- energy expectation value
  $$E[\kappa_1, ..., \kappa_\Xi] = \int d^3x\; \mathcal{E}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x})$$
- density
  $$\rho_\xi(\vec{x}) = \frac{1}{6\pi^2}\, \kappa_\xi^3(\vec{x})$$
- particle number
  $$N_\xi[\kappa_\xi] = \int d^3x\; \rho_\xi(\vec{x})$$

Ground State — Variationally

Functional Variation

minimization of the energy $E[\kappa_1, ..., \kappa_\Xi]$
for fixed numbers of particles $N_1, ..., N_\Xi$
gives the ground state density profile

- **chemical potentials**: implement constraints on the particle numbers via a set of Lagrange multipliers $\mu_1, ..., \mu_\Xi$
- unconstraint minimization of the transformed energy functional

$$F[\kappa_1, ..., \kappa_\Xi] = E[\kappa_1, ..., \kappa_\Xi] - \sum_{\xi=1}^{\Xi} \mu_\xi N_\xi[\kappa_\xi]$$
$$= \int \mathrm{d}^3x \, \mathcal{F}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x})$$

- stationary points of the energy density are solutions of the Euler-Lagrange equations

$$\frac{\partial}{\partial \kappa_\xi(\vec{x})} \mathcal{F}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x}) = 0 \,, \quad \forall \xi$$

- since $\mathcal{F}_\Xi$ is local (does not depend on gradients) the ground state has to minimize $\mathcal{F}_\Xi$ for each $\vec{x}$

Recipe

ground-state densities at some $\vec{x}$ are
given by the minimum of the transformed
energy density $\mathcal{F}[\kappa_1, ...\kappa_\Xi](\vec{x})$ for this $\vec{x}$

Structure of a Trapped Degenerate Two-Component Fermi Gas

- **Energy Landscapes & Density Profiles**
- **Mean-Field Induced Collapse**
- **Component Separation**
- **Phase Diagram**

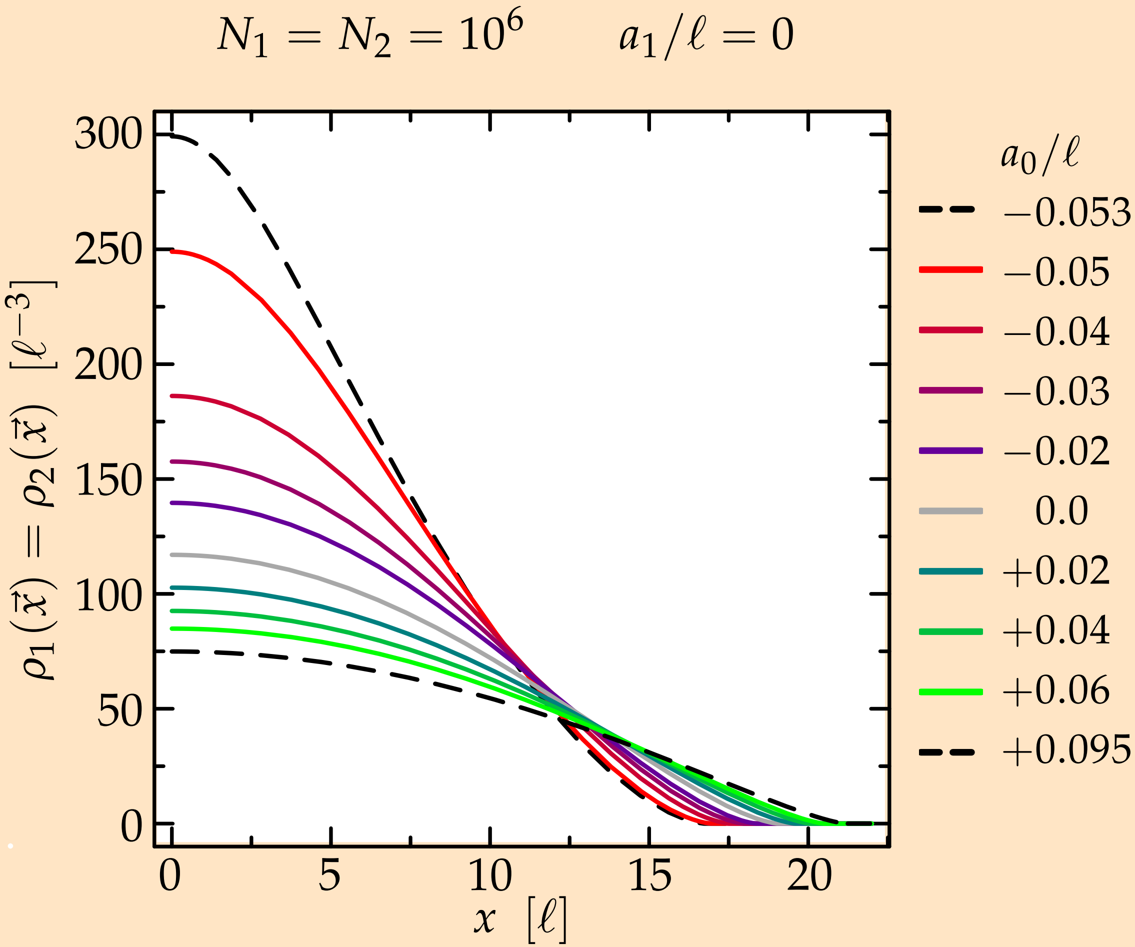
Density Profiles

- assume a spherical symmetric parabolic trapping potential

$$U(\vec{x}) = \frac{m\omega^2}{2}\,x^2 = \frac{1}{2m\ell^4}\,x^2$$

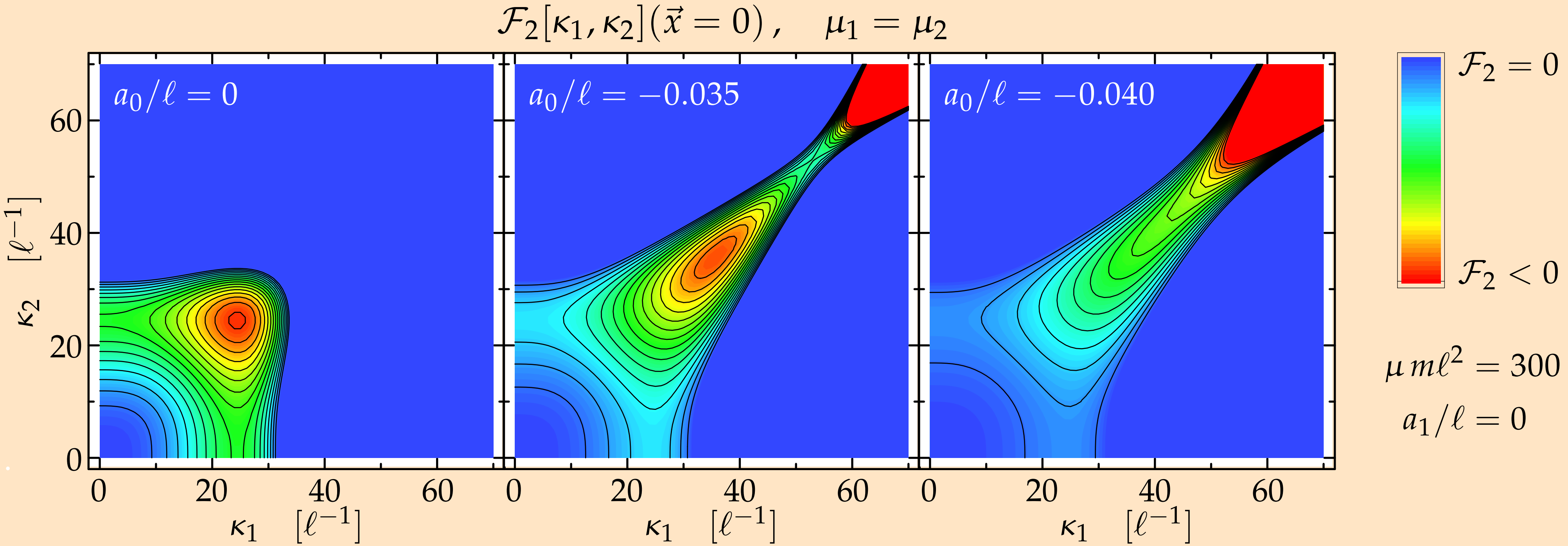
- determine the densities for μ_1, μ_2 chosen such that the desired particle numbers are reproduced

- $a_0 > 0$: repulsive interactions flatten the density profile
- $a_0 < 0$: attractive interactions enhance the central density
- outside a certain range of scattering lengths a_0 no solutions of this type exist anymore



Energy-Density Landscape: $a_0 < 0$

$\mathcal{F}_2[\kappa_1, \kappa_2](\vec{x}=0)$, $\mu_1 = \mu_2$

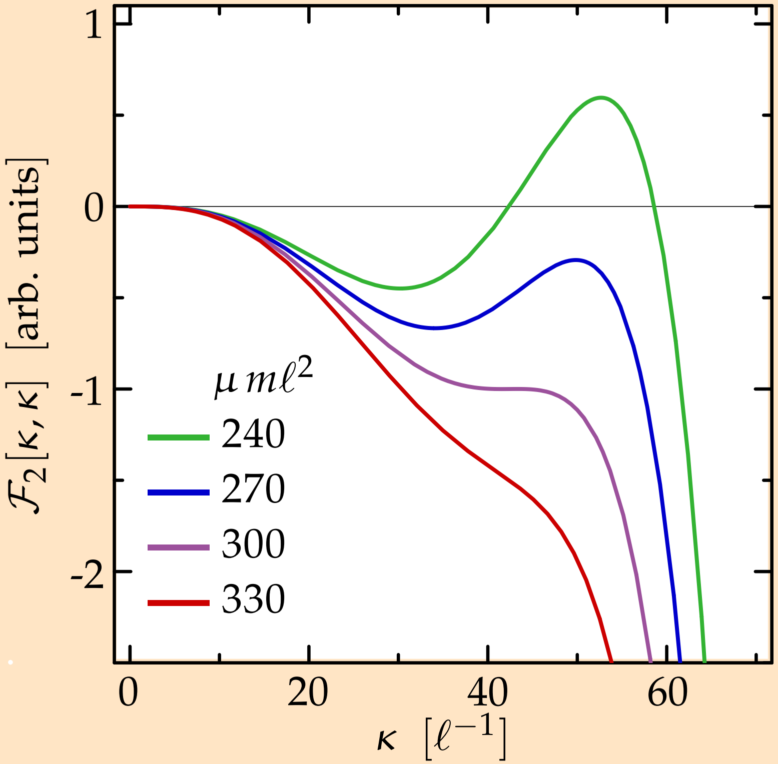


- minimum of $\mathcal{F}_2$ is only local for attractive interactions ($a_0 < 0$ or $a_1 < 0$)
- NB: physically the state is only metastable for all signs of the scattering lengths
- local minimum vanishes if the attractive s-wave interaction exceeds a critical strength

attractive interactions can induce a **collapse** of the Fermi gas towards high densities

Collapse — Stability Condition

$$\mathcal{F}_2[\kappa,\kappa](\vec{x}) = \frac{1}{3\pi^2}\left[U(\vec{x})-\mu\right]\kappa^3(\vec{x}) + \frac{1}{10\pi^2 m}\kappa^5(\vec{x}) + \frac{a_0}{9\pi^3 m}\kappa^6(\vec{x}) + \frac{a_1^3}{10\pi^3 m}\kappa^8(\vec{x})$$



$a_0/\ell = -0.037, \quad a_1/\ell = 0, \quad \vec{x} = 0$

- onset of instability is indicated by the appearance of a saddle point in the energy density, i.e., a vanishing first and second derivative
- **stability condition**: metastable states exist only for

  $\mu < \mu_{cr}(a_0, a_1)$ and $\kappa(\vec{x}) < \kappa_{cr}(a_0, a_1)$

- the critical Fermi momentum and the critical chemical potential are given by

$$-2\,a_0\,\kappa_{cr} - 4\,(a_1\,\kappa_{cr})^3 = \pi$$

$$m\,\mu_{cr} = \frac{1}{2}\,\kappa_{cr}^2 + \frac{2\,a_0}{3\pi}\,\kappa_{cr}^3 + \frac{8\,a_1^3}{15\pi}\,\kappa_{cr}^5$$

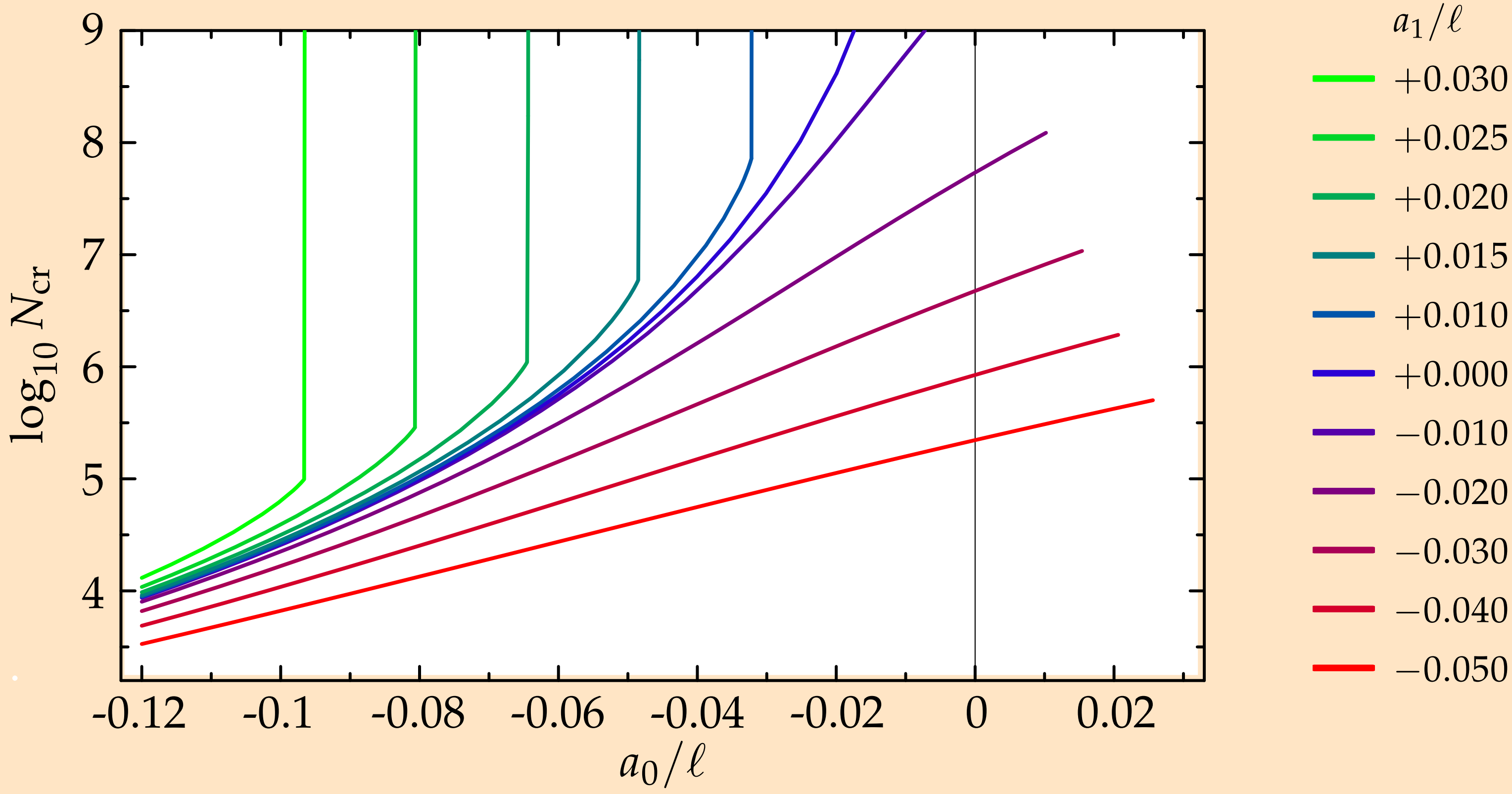
Collapse — Critical Particle Number

- assume parabolic trapping potential with mean oscillator length ℓ
- obtain the density profile for the critical chemical potential $\mu_{\rm cr}$ and calculate $N_{\rm cr}$

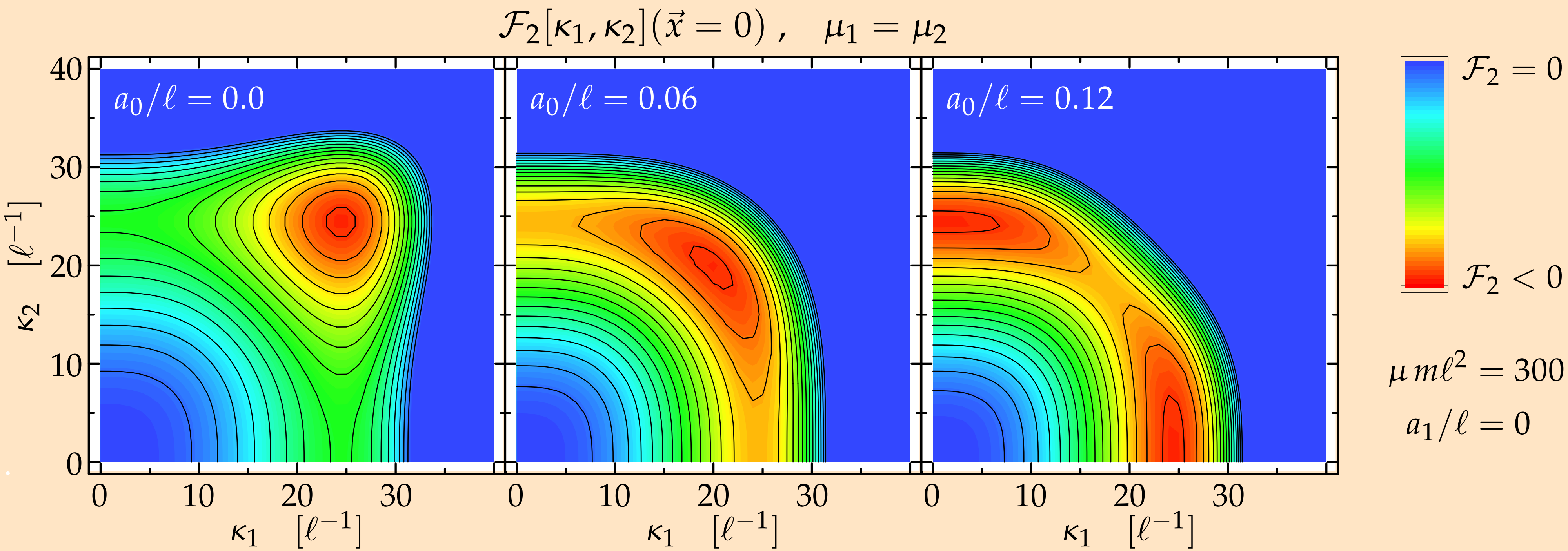
abs. stabilization due to p-wave repulsion $a_1/|a_0| > 2/(3\pi^{2/3})$

p-wave attraction lowers critical particle number substantially

p-wave induced collapse and interference with separation



Energy-Density Landscape: $a_0 > 0$



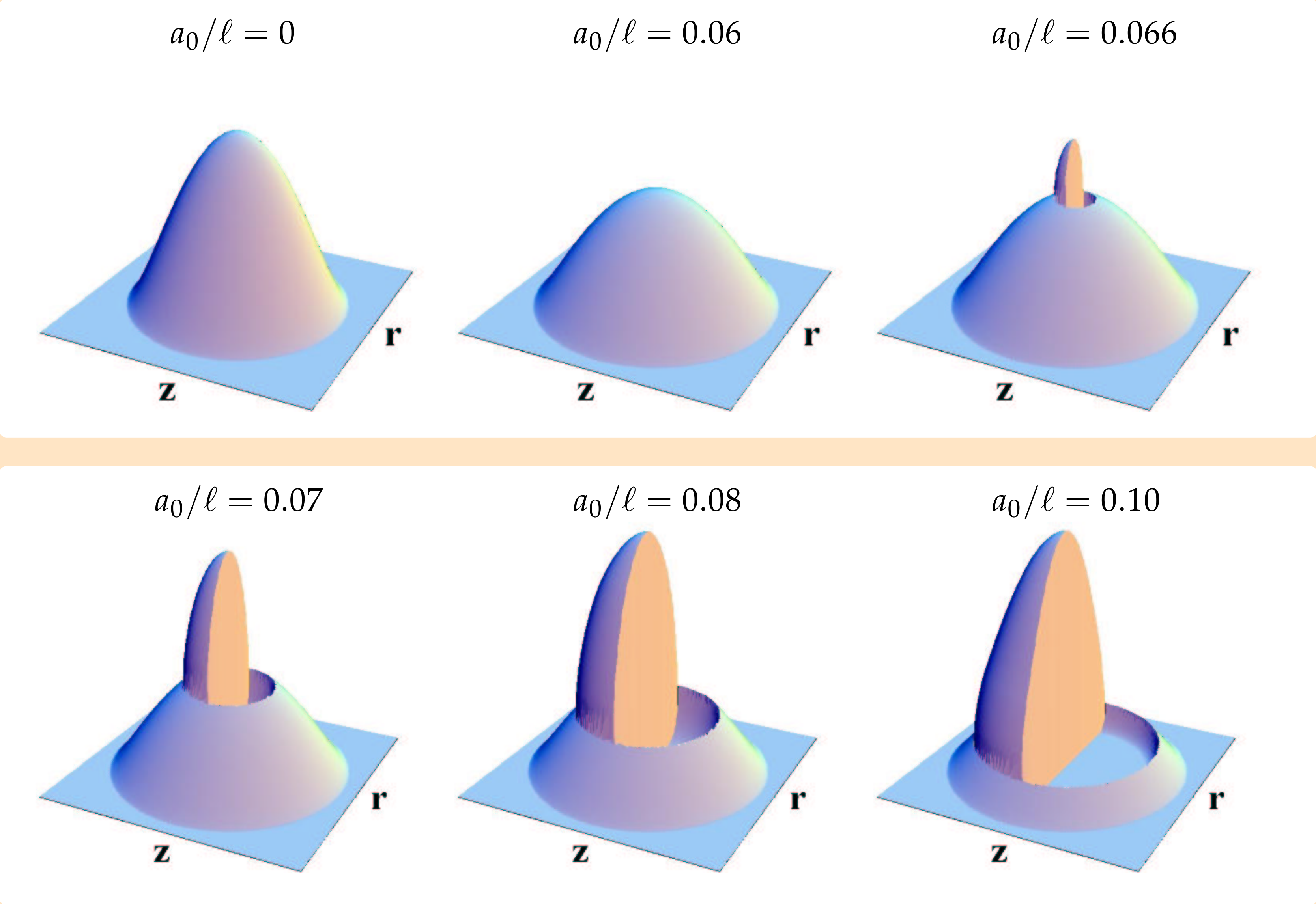
- **overlapping configuration**: for moderate repulsive s-wave interactions a unique minimum exists at $\kappa_1 = \kappa_2$
- **separation**: beyond a critical interaction strength two separate minima emerge at

$$\kappa_1 = 0,\ \kappa_2 > 0 \qquad \text{and} \qquad \kappa_1 > 0,\ \kappa_2 = 0$$

repulsive interactions can induce a spatial **separation** of the two components

Separation — Density Distributions

$$\rho_1(r,z) = \rho_2(r,-z)$$



$N_1 = N_2 = 10^7$

$a_1/\ell = 0$

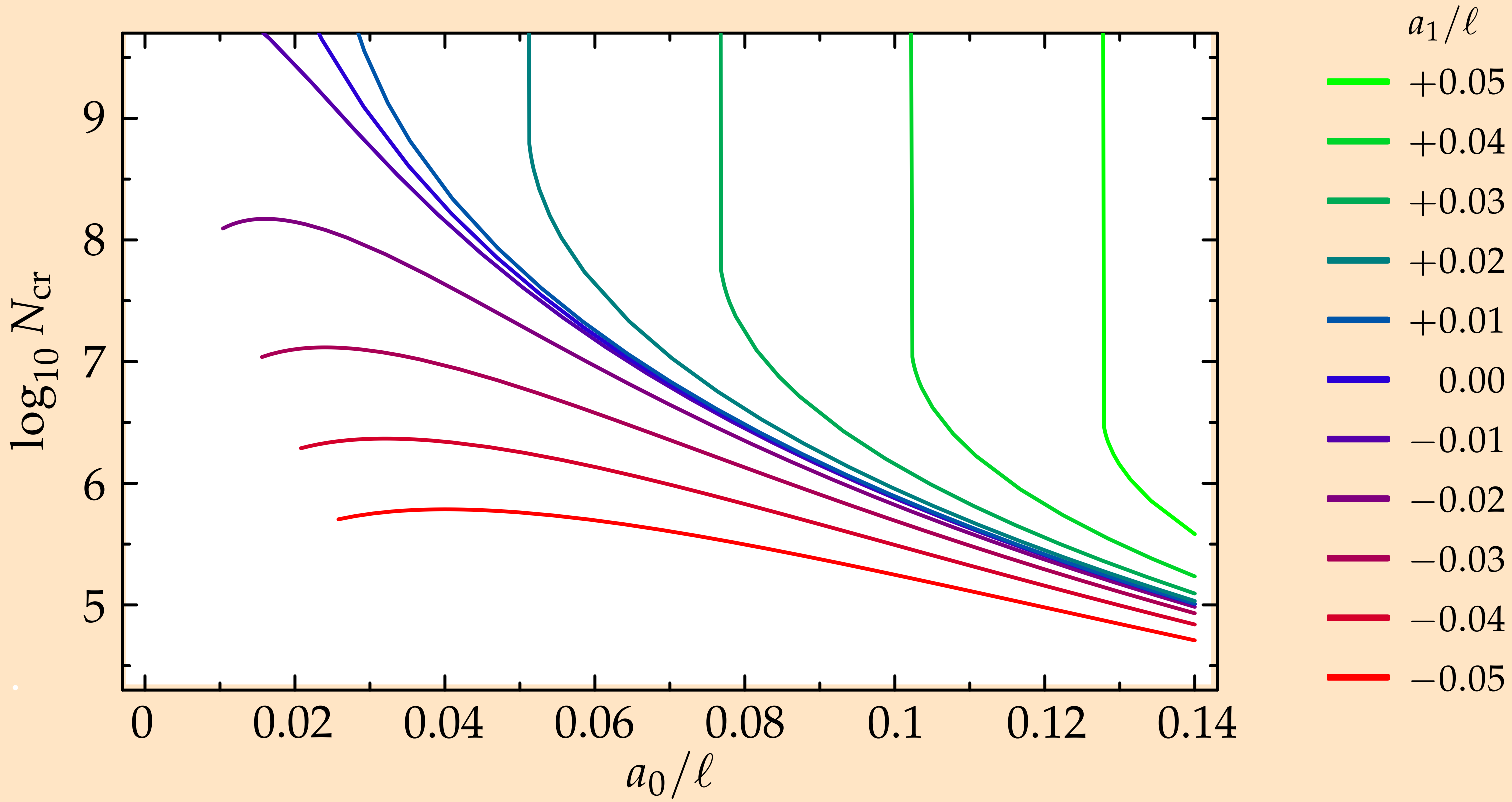
Separation — Critical Particle Number

- assume parabolic trapping potential with mean oscillator length ℓ
- obtain the density profile for the critical chemical potential μ_{cr} and calculate N_{cr}

interference with collapse induced by p-wave attraction

p-wave attraction lowers critical particle number substantially

abs. stabilization due to p-wave repulsion $a_1/a_0 > 2^{4/3}/(3\pi^{2/3})$

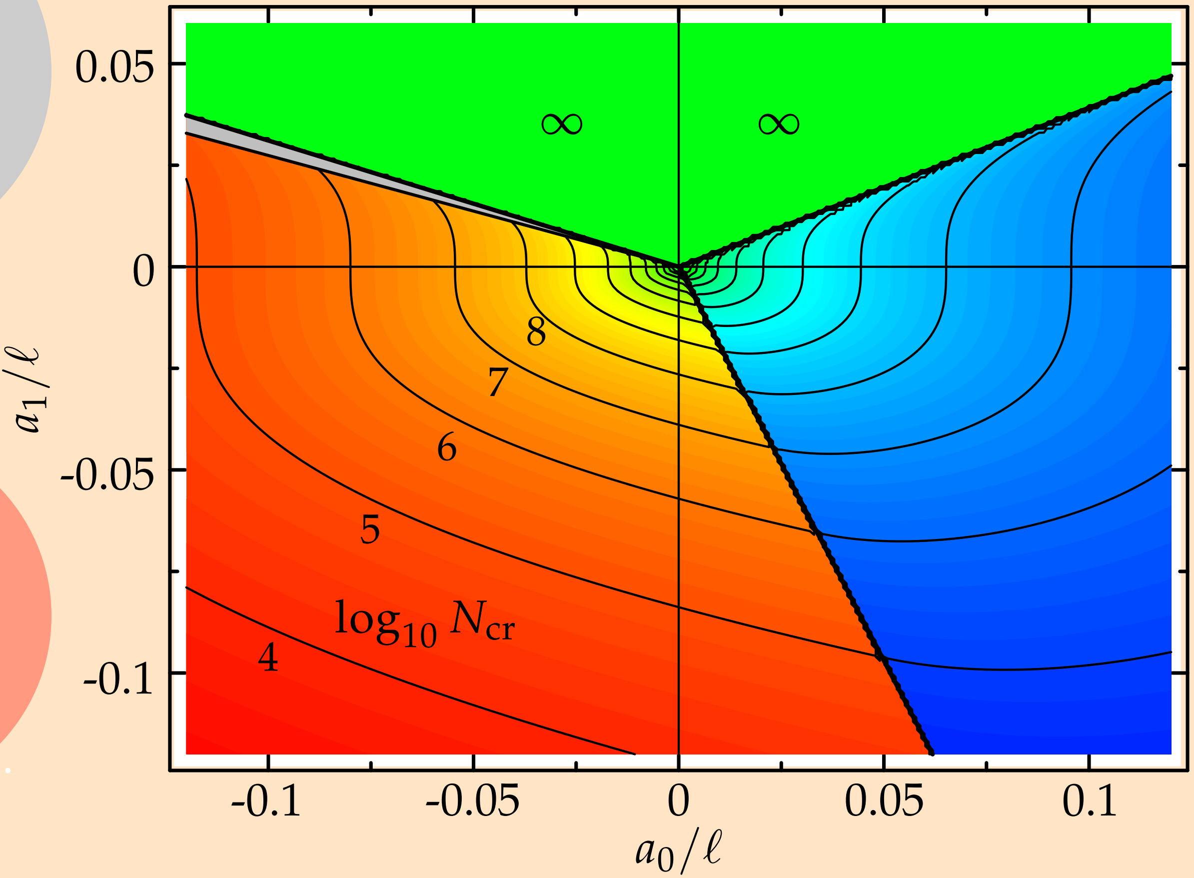


Two-Component Fermi Gas

Stability Map

overlapping conf. is stable for all particle numbers

p-wave stabilized high-density phase above N_{cr}



components separate above critical particle number

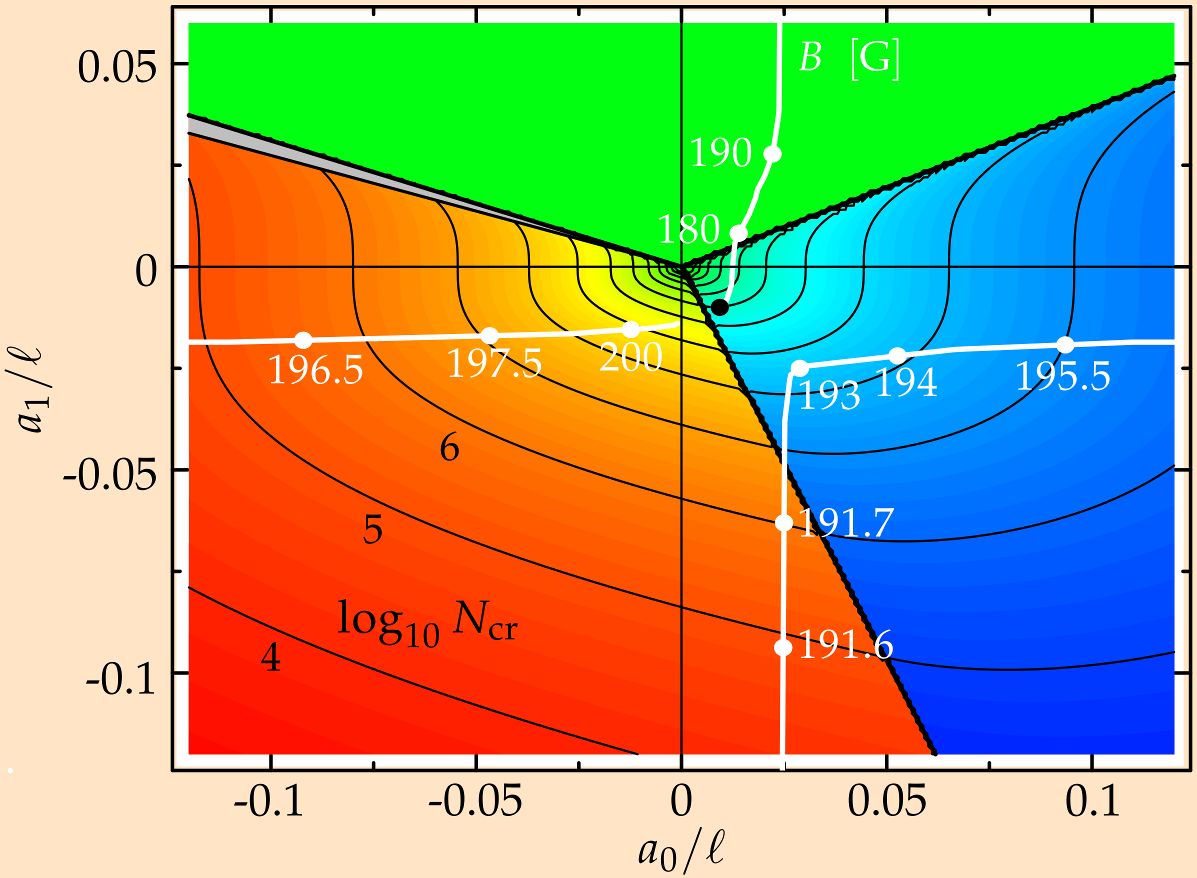
mean-field collapse above critical particle number

Stability Map & Feshbach Resonances

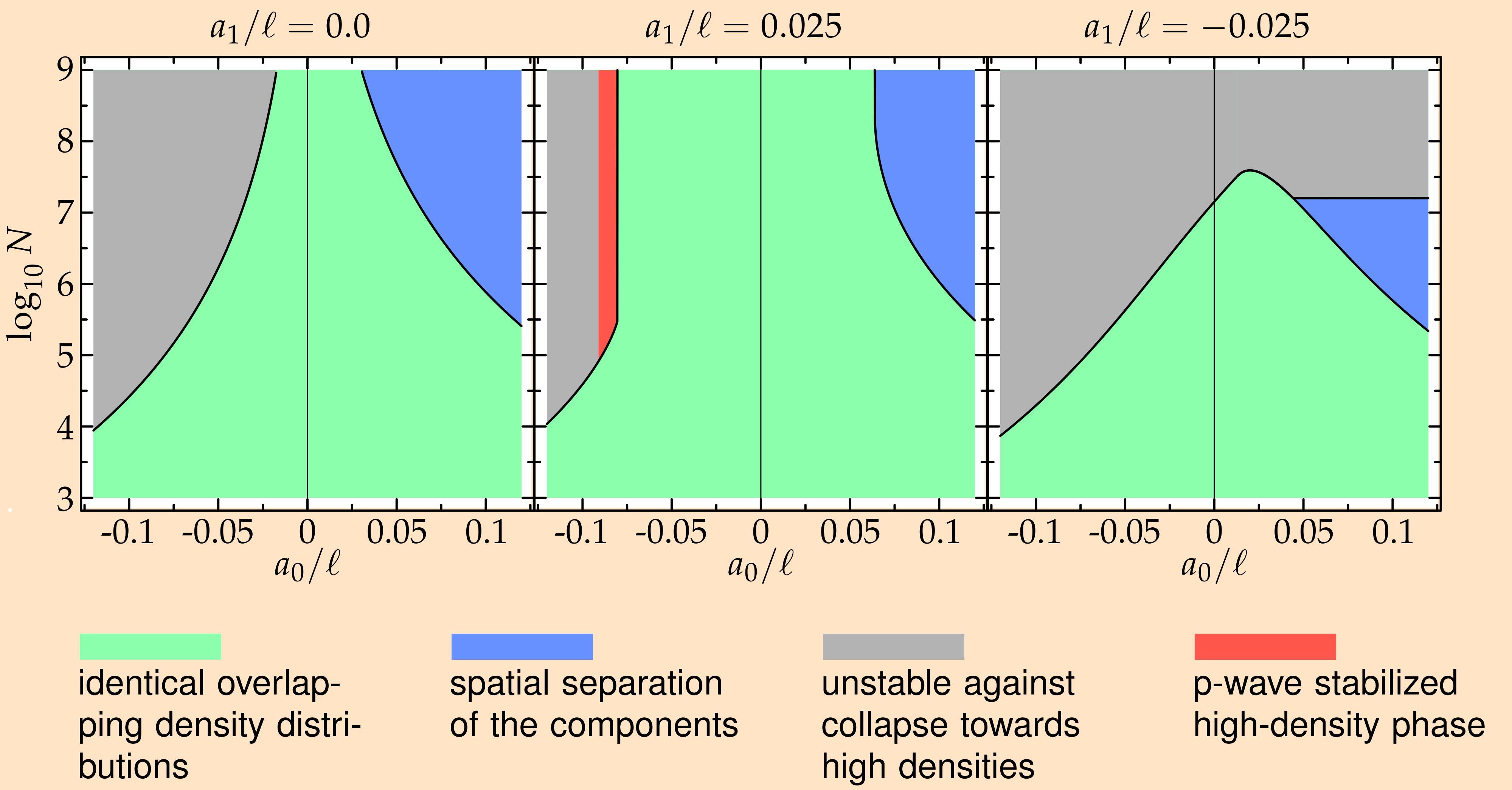
- **Feshbach resonances** allow to tune the strength of the atom-atom interaction (scattering lengths) via an external magnetic field

- simultaneous s- and p-wave Feshbach resonance predicted for a two-component ^{40}K system with $F = \frac{9}{2}$, $m_F = -\frac{9}{2}, -\frac{7}{2}$

  J. Bohn, Phys. Rev. A61 (2000) 053409



Phase Diagram



Summary

Strategy

- developed a simple framework to describe interacting degenerate quantum gases
- effective contact interaction + mean-field states + Thomas-Fermi approximation → energy functional
- investigated the influence of s- and p-wave interactions on structure and stability of degenerate Fermi gases

Results

- s- and p-wave interactions have strong influence on the density profiles and the stability of the gas
- **collapse**: attractive interactions can induce a collapse of the dilute gas towards high densities
- **separation**: repulsive interactions can cause a spatial separation of the different components
- in all cases a complex interplay between s- and p-wave interactions is observed

...have a look at

http://theory.gsi.de/˜trap